Section 2.2: Vectors and Dot Product

- Review on Planar Vectors

A 2D vector is a line segment with a direction. Two vectors are considered equivalent if they have the same length and same direction.

In the figure, all three vectors are equivalent. Equivalent vectors are treated as the same. The vector originated from the origin is the representative of the equivalent class it belongs, and is denoted by $\langle x, y \rangle$, where $(x, y)$ is the coordinate of the end point of this representative. In particular, if $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points, then the vector connecting $A$ & $B$ pointing from $A$ to $B$ is denoted by $\overrightarrow{AB}$, and has the representative originated from the origin and endpoint $(x_2 - x_1, y_2 - y_1)$. Vector addition, subtraction, and scalar multiplication are defined as

\[
\langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle \\
\langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle = \langle x_1 - x_2, y_1 - y_2 \rangle \\
\lambda \langle x_1, y_1 \rangle = \langle \lambda x_1, \lambda y_1 \rangle.
\]

The length of vector $< x_1, y_1 >$ is

\[
|\langle x_1, y_1 \rangle| = \sqrt{x_1^2 + y_1^2}.
\]

Geometrically, let $\vec{u}$ and $\vec{v}$ be two vectors. Then $\vec{u} + \vec{v}$ is the diagonal vector of the parallelogram formed by $\vec{u}$ and $\vec{v}$:
If $\lambda > 0$, $\lambda \vec{u}$ is the vector with the same direction as $\vec{u}$ and with length $\lambda$ times the length of $\vec{u}$, i.e.,

$$|\lambda \vec{u}| = \lambda |\vec{u}|.$$

If $\lambda < 0$, $\lambda \vec{u}$ is the vector with the opposite direction to $\vec{u}$ and with length $|\lambda|$ times the length of $\vec{u}$, i.e.

$$|\lambda \vec{u}| = |\lambda| |\vec{u}|.$$

In particular,

$$\left(\frac{1}{|\vec{u}|}\right) \vec{u}$$

is the vector of unit length have the same direction as $\vec{u}$, and is called the unit vector in the direction of $\vec{u}$, or simply the **direction of** $\vec{u}$.

**Vectors in 3D Space**

**Definition:** A vector in $R^3$ is a line segment in 3D space with a direction. Two vectors are considered equivalent if they have the same length and same direction. The vector originated from the origin is the representative of the equivalent class it belongs, and is denoted by $< x, y, z >$, where $(x, y, z)$ is the coordinate of the end point of this representative. In particular, if $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points, then the vector connecting $A$ & $B$ pointing from $A$ to $B$ is denoted by $\overrightarrow{AB}$, and has the representative

$$\overrightarrow{AB} = < x_2 - x_1, y_2 - y_1, z_2 - z_1 > \quad (= \langle B \rangle - \langle A \rangle )$$
In the above figure, all three vectors are equivalent, and thus are considered equal. Note that a vector consists of two elements: length and direction. Two vectors having the same length and direction are equal.

**Example 2.1.** Find $\overrightarrow{AB}$ if $A (2, -3, 4)$ and $B (-2, 1, 1)$.

Solution:

$$\overrightarrow{AB} = \langle -2 - 2, 1 + 3, 1 - 4 \rangle = \langle -4, 4, -3 \rangle.$$  

Just like 2D vectors, for 3D vector addition, subtraction, and scalar multiplication are defined similarly.

- **Vector Addition/Subtraction:** Given $\vec{u} = \langle x_1, y_1, z_1 \rangle$, $\vec{v} = \langle x_2, y_2, z_2 \rangle$. Then

  $$\vec{u} + \vec{v} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

  $$\vec{u} - \vec{v} = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle.$$  

- **Scalar Multiplication:** $\lambda$ is a constant.

  $$\lambda \vec{u} = \lambda \langle x_1, y_1, z_1 \rangle = \langle \lambda x_1, \lambda y_1, \lambda z_1 \rangle.$$  

- **Length of $\vec{u}$:**

  $$|\vec{u}| = \sqrt{(x_1)^2 + (y_1)^2 + (z_1)^2}.$$
• Unit vector in the direction of $\vec{u}$:

$$\frac{1}{|\vec{u}|}\vec{u} = \left\langle \frac{x_1}{\sqrt{(x_1)^2 + (y_1)^2 + (z_1)^2}}, \frac{y_1}{\sqrt{(x_1)^2 + (y_1)^2 + (z_1)^2}}, \frac{z_1}{\sqrt{(x_1)^2 + (y_1)^2 + (z_1)^2}} \right\rangle.$$

**Example 2.2.** Find $\vec{u} + \vec{v}$, $2\vec{u} - \vec{v}$, $|\vec{u}|$, and the unit vector in the direction of $\vec{u}$, if $\vec{u} = \langle 2, -3, 4 \rangle$ and $\vec{v} = \langle -2, 1, 1 \rangle$.

Solution:

$$\vec{u} + \vec{v} = \langle 2, -3, 4 \rangle + \langle -2, 1, 1 \rangle = \langle 0, -2, 5 \rangle$$

$$2\vec{u} = 2 \langle 2, -3, 4 \rangle = \langle 4, -6, 8 \rangle$$

$$2\vec{u} - \vec{v} = \langle 4, -6, 8 \rangle - \langle -2, 1, 1 \rangle = \langle 6, -7, 7 \rangle$$

$$|\vec{u}| = \sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{29}$$

$$\frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{29}} \langle 2, -3, 4 \rangle = \left\langle \frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \right\rangle.$$

• **Property:** $\vec{u}$, $\vec{v}$ and $\vec{w}$ are three vectors, $\lambda$ and $\delta$ are two real numbers. Then

(1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(2) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

(3) $\vec{u} + \vec{0} = \vec{u}$

(4) $\vec{u} + (-\vec{u}) = \vec{0}$

(5) $\lambda (\vec{u} + \vec{v}) = \lambda \vec{u} + \lambda \vec{v}$

(6) $(\lambda + \delta) \vec{u} = \lambda \vec{u} + \delta \vec{u}$

(7) $(\lambda \delta) \vec{u} = \lambda (\delta \vec{u})$

(8) $1 \cdot \vec{u} = \vec{u}$

All these properties can be verified by direct computation. For instance, set $\vec{u} = \langle x_1, y_1, z_1 \rangle$, $\vec{v} = \langle x_2, y_2, z_2 \rangle$. Then

$$\vec{u} + \vec{v} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

$$\vec{v} + \vec{u} = \langle x_2, y_2, z_2 \rangle + \langle x_1, y_1, z_1 \rangle = \langle x_2 + x_1, y_2 + y_1, z_2 + z_1 \rangle$$

$$\Longrightarrow \vec{u} + \vec{v} = \vec{v} + \vec{u}. \text{ (Property (1))}.$$
• Base Vector Representation: Unit vectors in directions of axes are called base vectors. There are three base vectors:

\[ \vec{i} = (1, 0, 0) \quad \text{(unit vector in } x - \text{axis direction)} \]
\[ \vec{j} = (0, 1, 0) \quad \text{(unit vector in } y - \text{axis direction)} \]
\[ \vec{k} = (0, 0, 1) \quad \text{(unit vector in } z - \text{axis direction).} \]

Any vector \( \vec{u} = < x_1, y_1, z_1 > \) can be expressed as a linear combination of base vectors:

\[ \vec{u} = \langle x_1, y_1, z_1 \rangle = x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k}. \]

**Example 2.3.** Let \( \vec{u} = \vec{i} + 2\vec{j} - 3\vec{k}, \ \vec{v} = 4\vec{i} + 7\vec{k}. \) Find \( 2\vec{u} + 3\vec{v} \) and the unit vector in the direction of \( \vec{u}. \)

**Solution:**

\[ 2\vec{u} + 3\vec{v} = 2 (\vec{i} + 2\vec{j} - 3\vec{k}) + 3 (4\vec{i} + 7\vec{k}) \]
\[ = 2\vec{i} + 4\vec{j} - 6\vec{k} + 12\vec{i} + 21\vec{k} \]
\[ = 14\vec{i} + 4\vec{j} + 15\vec{k}. \]

\[ \frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{1 + 4 + 9}} \vec{u} = \frac{1}{\sqrt{14}} (\vec{i} + 2\vec{j} - 3\vec{k}) = \frac{\vec{i}}{\sqrt{14}} + \frac{2}{\sqrt{14}} \vec{j} - \frac{3}{\sqrt{14}} \vec{k}. \]

**The Dot Product**

**Definition:** The dot product of two vectors \( \vec{u} \) and \( \vec{v}, \) denoted by \( \vec{u} \cdot \vec{v}, \) is defined as

\[ \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta, \quad \theta = < \vec{u}, \vec{v} > \text{ is the angle between } \vec{u} \text{ and } \vec{v}, \quad 0 \leq \theta \leq \pi. \]

(1)

The dot product is also called Inner Product or Scalar product.

**Example 2.4.** Let \( |\vec{u}| = 4, \ |\vec{v}| = 6, \ \theta = \pi/3. \) Find \( \vec{u} \cdot \vec{v}. \)

**Solution:**

\[ \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta = 4 \cdot 6 \cdot \cos \frac{\pi}{3} = 12. \]
In the right triangle on the left of above figure,

\[ |\vec{u}| \cos \theta = \text{length of adjacent edge (with respect to } \theta). \]

If \( \vec{u} \) represents a vector of force, then \( |\vec{u}| \cos \theta \) is the horizontal component of the force. Therefore, if \( \vec{u} \) is a (constant) force vector dragging an object causing the object moving distance \( \vec{v} \) (i.e., in the direction of \( \vec{v} \) by distance \( |\vec{v}| \).) Then,

work done = (component of force in the direction of \( \vec{v} \) \( \times \) (distance \( |\vec{v}| \))

\[ = (|\vec{u}| \cos \theta) |\vec{v}| \]

\[ = \vec{u} \cdot \vec{v} \]

**Example 2.5.** A crate is hauled 8 meter up a ramp a force of 200 Newton at \( 25^\circ \) angle. Find work done.

Solution: Let \( \vec{F} \) be the force vector and \( \vec{r} \) is the distance vector. Then

work done = \( \vec{F} \cdot \vec{r} = |\vec{F}| |\vec{r}| \cos \theta \)

\[ = 200 \times 8 \times \cos (25^\circ) \]

\[ = 160 \cos (25^\circ). \]
**Theorem.** (a) Two vectors $\vec{u}$ and $\vec{v}$ are perpendicular (denoting by $\vec{u} \perp \vec{v}$) iff

$$\vec{u} \cdot \vec{v} = 0.$$ 

(b) 

$$\vec{u} \cdot \vec{u} = |\vec{u}|^2, \text{ or } |\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$$

Proof: (a) Recall Definition (1) that

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta.$$ 

So $\vec{u} \cdot \vec{v} = 0$ iff $\cos \theta = 0$. Now, $\vec{u} \perp \vec{v} \iff \theta = \frac{\pi}{2} \iff \cos \theta = 0$. (b) When $\vec{u}$ and $\vec{v}$ have the same direction, in particular, when $\vec{u} = \vec{v}$, we have $\theta = 0 \implies \cos \theta = 1$.

The definition of the dot product is not convenient for calculation.

**Component Form of Dot Product:** Let $\vec{u} = <x_1, y_1, z_1>$, $\vec{v} = <x_2, y_2, z_2>$. Then

$$\vec{u} \cdot \vec{v} = x_1 x_2 + y_1 y_2 + z_1 z_2.$$  \hspace{1cm} (2)

Proof: We shall use the Law of Cosine for triangles studied in pre-calculus

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

to verify (2), where $a, b, c$ are lengths of three edges and $\theta$ is the angle opposite to $c$.

If we use vector notations as in the above figure, then

$$a = |\vec{u}|, \ b = |\vec{v}|, \ c = |\vec{u} - \vec{v}|,$$

and thus the Law of Cosine becomes

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2 |\vec{u}| |\vec{v}| \cos \theta.$$
Using the definition of the dot product (1), i.e.,
\[ \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta, \]
the Law of Cosine has the vector form
\[ |\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2 \vec{u} \cdot \vec{v}. \] (3)

On the other hand,
\[ |\vec{u}|^2 = x_1^2 + y_1^2 + z_1^2 \]
\[ |\vec{v}|^2 = x_2^2 + y_2^2 + z_2^2 \]
and thus
\[ |\vec{u} - \vec{v}|^2 = |(x_1 - x_2, y_1 - y_2, z_1 - z_2)|^2 \]
\[ = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \]
\[ = (x_1^2 - 2x_1x_2 + x_2^2) + (y_1^2 - 2y_1y_2 + y_2^2) + (z_1^2 - 2z_1z_2 + z_2^2) \]
\[ = x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2 - 2(x_1x_2 + y_1y_2 + z_1z_2) \]
\[ = |\vec{u}|^2 + |\vec{v}|^2 - 2(x_1x_2 + y_1y_2 + z_1z_2). \]

It follows from (3) that
\[ |\vec{u}|^2 + |\vec{v}|^2 - 2 \vec{u} \cdot \vec{v} \]
\[ = |\vec{u}|^2 + |\vec{v}|^2 - 2(x_1x_2 + y_1y_2 + z_1z_2), \]
from which we easily derive at
\[ \vec{u} \cdot \vec{v} = x_1x_2 + y_1y_2 + z_1z_2. \]

**Example 2.6.** Let \( \vec{u} = \langle 2, 4, 1 \rangle, \quad \vec{v} = \langle 3, -1, 2 \rangle \). Find (a) \( \vec{u} \cdot \vec{v} \) and (b) the angle between \( \vec{u} \) and \( \vec{v} \).

**Solution:** (a)
\( \vec{u} \cdot \vec{v} = \langle 2, 4, 1 \rangle \cdot \langle 3, -1, 2 \rangle \)
\[ = 2 \times 3 + 4 \times (-1) + 1 \times 2 = 4. \]

(b) In formula (9.3.1), dividing both sides by \( |\vec{u}| |\vec{v}| \) would lead to
\[ \cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} = \frac{4}{\sqrt{4 + 16} + 1\sqrt{9 + 1} + 4} = \frac{4}{\sqrt{21} \sqrt{14}} = 0.233. \]
So
\[ \theta = \arccos (0.233) = 1.3356 \text{ (radian)} \]
\[ = 1.3356 \times \frac{180}{\pi} \text{ (deg)} = 57.296^\circ. \]

**Example 2.7.** Show that
\[ \left( 2\vec{i} + 2\vec{j} - \vec{k} \right) \perp \left( 5\vec{i} - 4\vec{j} + 2\vec{k} \right). \]

Solution: We know that two vectors are perpendicular iff their dot product is zero. Now since
\[ \left( 2\vec{i} + 2\vec{j} - \vec{k} \right) \cdot \left( 5\vec{i} - 4\vec{j} + 2\vec{k} \right) = 2 \times 5 + 2 \times (-4) + (-1) \times 2 = 0 \]
we proved that these two vectors are orthogonal to each other.

**Some Properties of Dot Product:**
1. \( \vec{u} \cdot \vec{u} = |\vec{u}|^2 \)
2. \( \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \)
3. \( \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \)
4. \( (\lambda \vec{u}) \cdot \vec{v} = \lambda (\vec{u} \cdot \vec{v}) = \vec{u} \cdot (\lambda \vec{v}) \)
5. \( \vec{0} \cdot \vec{u} = \vec{u} \cdot \vec{0} = 0. \)

**Orthogonal Projection of** \( \vec{u} \) **onto** \( \vec{v} \) **is the vector with length** \( |\vec{u}| \cos \theta \) **having either the same direction as** \( \vec{v} \) **(if** \( \theta \leq \pi/2 \) **) or opposite to** \( \vec{v} \) **(if** \( \theta > \pi/2 \), **and is denoted by** \( \text{Proj}_v(\vec{u}) \):**
\[ \text{Proj}_v(\vec{u}) = (|\vec{u}| \cos \theta) \frac{\vec{v}}{|\vec{v}|} = \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}. \]
When \( \vec{v} \) is a unit vector, i.e., \( |\vec{v}| = 1 \), we have
\[
\text{Proj}_{\vec{v}}(\vec{u}) = (\vec{u} \cdot \vec{v}) \vec{v}.
\]
Note that \( \text{Proj}_{\vec{v}}(\vec{u}) \) depends only on the directions parallel to \( \vec{v} \). In other words,
\[
\text{Proj}_{\vec{v}}(\vec{u}) = \text{Proj}_{\vec{v}}(\vec{u}) \quad \text{if} \quad \vec{v} = \lambda \vec{w} \text{ (i.e., if } \vec{v} \text{ and } \vec{w} \text{ are parallel)}.
\]

**Example 2.5.** Let \( \vec{u} = <-2, 3, 1> \), \( \vec{v} = <1, 1, 2> \). Find \( \text{Proj}_{\vec{v}}(\vec{u}) \).
Solution:
\[
\text{Proj}_{\vec{v}}(\vec{u}) = \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v} = \frac{-2 + 3 + 2}{(1+1+4)} <1, 1, 2>
= \frac{3}{6} <1, 1, 2> = \frac{1}{2} <1, 1, 2>.
\]

Homework for Section 2.2:

1. Vectors \( \vec{a} \) and \( \vec{b} \) are shown below. Draw the following vectors:
   
   (a) \( \vec{a} + 2\vec{b} \)
   (b) \( \vec{a} - 2\vec{b} \)
   (c) \( \vec{b} - 3\vec{a} \)
   (d) \( 3\vec{a} - 2\vec{b} \)
2. Find vector $\overrightarrow{AB}$ if

(a) $A(2, 4, -1), B(1, -1, 2)$
(b) $A(-2, 3, 0), B(4, 2, -5)$

3. Find $\vec{a} + 2\vec{b}, -3\vec{a}, |\vec{a}|, |\vec{a} + 2\vec{b}|$, and a unit vector with the same direction as $\vec{a}$

(a) $\vec{a} = \langle 3, 2, 1 \rangle, \quad \vec{b} = \langle 2, 1, -2 \rangle$
(b) $\vec{a} = \langle -1, 0, 2 \rangle, \quad \vec{b} = \langle 1, -2, 2 \rangle$

4. Using vector notation, determine whether three points lie on straight line.

(a) $A(2, 4, 2), B(3, 7, -2), C(1, 3, 3)$
(b) $A(1, -4, -4), B(2, -1, 5), C(4, 5, 3)$

5. Which of the following expressions are meaningful?

(a) $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$
(b) $(\vec{a} \cdot \vec{b}) \vec{c}$
(c) $(\vec{a} \cdot \vec{b}) + \vec{c}$
(d) $(\vec{a} + \vec{b}) \cdot \vec{c}$
(e) $(\vec{a} + \vec{b}) \cdot |\vec{c}|$
(f) \( \left( \vec{a} + \left| \vec{b} \right| \right) \cdot \vec{c} \)

6. Find \( \vec{a} \cdot \vec{b} \).

(a) \( |\vec{a}| = 2, |\vec{b}| = 3 \), and the angle between them is \( \pi/3 \)

(b) \( \vec{a} = \langle 2s, s, -4s \rangle, \quad \vec{b} = \langle -3s, 2s, 2s \rangle \)

7. Find \( \vec{a} \cdot \vec{b} \) and the angle between them.

(a) \( \vec{a} = \langle 1, 2, -3 \rangle, \quad \vec{b} = \langle -2, 1, 4 \rangle \)

(b) \( \vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}, \quad \vec{b} = -\vec{i} - 4\vec{j} + 2\vec{k} \)

8. Find the orthogonal projection of \( \vec{a} \) onto \( \vec{b} \). (Notice that \( \vec{a} \) and \( \vec{b} \) are the same as in #3)

(a) \( \vec{a} = \langle 1, 2, -3 \rangle, \quad \vec{b} = \langle -2, 1, 4 \rangle \)

(b) \( \vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}, \quad \vec{b} = -\vec{i} - 4\vec{j} + 2\vec{k} \)

9. Determine whether \( \vec{a} \) and \( \vec{b} \) are orthogonal.

(a) \( \vec{a} = \langle 1, 2, -3 \rangle, \quad \vec{b} = \langle 0, 3, 2 \rangle \)

(b) \( \vec{a} = \langle 2, -6, -4 \rangle, \quad \vec{b} = \langle -7, -9, 10 \rangle \)

(c) \( \vec{a} = -2\vec{i} - 3\vec{j} + 4\vec{k}, \quad \vec{b} = -\vec{i} + 4\vec{j} + 2\vec{k} \)

10. Find a unit vector that is orthogonal to both \( \vec{i} + \vec{j} \) and \( \vec{i} + \vec{k} \).

11. Find the work done by a force of 30 lb acting in the direction N 45° W in moving an object 5 ft west.