Chapter 1. Infinite Series
Section 1.1 Infinite Sequence

Definition of Sequences: A (infinite) sequence is a list of infinite many numbers in a pre-defined order. For instance,

\[ 1, 2, 3, 4, 5, 6, 7, 8, ... \]
\[ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, ... \]
\[ 2, 3, 4, 5, 6 \]
\[ \frac{2}{1}, -\frac{3}{2}, \frac{4}{3}, -\frac{5}{4}, ... \]

Note that since it is impossible to list infinite many numbers, we may stop at a certain point and omit the rest as long as a pattern is clear. For instance, the above sequences could be written as

\[ 1, 2, 3, 4, ..., \quad n \quad (n^{th} \text{ position}), ... \]
\[ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \quad \frac{1}{n} \quad (n^{th} \text{ position}), ... \]
\[ 2, 3, 4, 5, 6 \quad (n^{th} \text{ position}) \]
\[ \frac{2}{1}, -\frac{3}{2}, \frac{4}{3}, -\frac{5}{4}, ... \quad (\text{for } n+1) \quad \frac{n+1}{n} \quad (n^{th} \text{ position}) \]
In general, a sequence can be expressed as
\[ a_1, a_2, a_3, a_4, ..., a_n, ... \] or simply \( \{a_n\}_{n=1}^{\infty} \)
a_1 is called the first term,
a_2 is called the second term ...
a_n is called the nth term
\( a_{n+1} \) is the term right after \( a_n \).
a_{n-1} is the term right before \( a_n \).

Sometimes, for convenience, a sequence can be written as
\( \{b_n\}_{n=4}^{\infty} : b_4, b_5, b_6, ... \)
or \( \{c_n\}_{n=-3}^{\infty} : c_{-3}, c_{-2}, c_{-1}, c_0, c_1, ... \)

In many cases, a sequence \( a_n \) can be treated as a function defined for integer numbers, \( a_n = f(n) \).

**Example 1.1.** The three sequences introduced in the beginning,

1. 2, 3, 4, 5, 6, 7, 8, ...
2. \( 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, ... \)
3. \( 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, ... \)

can be defined by, respectively,

\[ a_n = f(n) = n, \ n = 1, 2, 3, ... \] or \( \{n\}_{n=1}^{\infty} \)
\[ a_n = f(n) = \frac{1}{n}, \ n = 1, 2, 3, ... \] or \( \left\{ \frac{1}{n} \right\}_{n=1}^{\infty} \)
\[ a_n = f(n) = (-1)^{n+1} \frac{n+1}{n}, \ n = 1, 2, 3, ... \] or \( \left\{ (-1)^{n+1} \frac{n+1}{n} \right\}_{n=1}^{\infty} \).

**Example 1.2.** Some sequences defined by formulas:

\( \{a_n\}_{n=0}^{\infty} = \left\{ \frac{n+1}{3^n} \right\}_{n=0}^{\infty} : \frac{1}{3^0}, \frac{2}{3^1}, \frac{3}{3^2}, \frac{4}{3^3}, ... \)

\( \{a_n\}_{n=0}^{\infty} = \left\{ \cos \frac{n\pi}{6} \right\}_{n=0}^{\infty} \)
\[ \{a_n\}_{n=3}^\infty = \{\sqrt{n - 3}\}_{n=3}^\infty : 0, \sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots \]

The last sequence has another equivalent expression:

\[ \{a_n\}_{n=0}^\infty = \{\sqrt{n}\}_{n=0}^\infty \]

**Example 1.3.** Some sequences may be defined by verbal descriptions:

\( a_n \) is the population of the world as of January 1st, year \( n \)

For instance, \( a_{2008} \) is the population of the world as of January 1st, 2008.

**Example 1.4.** Some sequences may be defined recursively:

(a) Fibonacci sequences: \( f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2}, \) for \( n = 2, 3, \ldots \)

\[ 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \]

(b) Average sequence: \( a_0 = 0, a_1 = 1, a_n = \frac{a_{n-1} + a_{n-2}}{2}, \) for \( n = 2, 3, \ldots \)

\[ 0, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}, \frac{21}{32}, \ldots \]

**Graph of Sequence**

The graph of a sequence \( \{a_n\}_{n=0}^\infty \) consists of a sequence of infinite many points \( \{(n, a_n)\}_{n=0}^\infty : \)
In this example, \( y = 1 \) is the "asymptote" or limit of the sequence.

If a sequence is defined through a function

\[
a_n = f(n),
\]

then the relation between the graph of the function \( y = f(x) \) and the graph of the sequence \( \{a_n\}_{n=0}^{\infty} \) is demonstrated in the following figure:

solid line — graph of function \( y = f(x) \); dots — graph of sequence \( \{a_n\} \)

**Limit of Sequence**

Since sequences consist of infinite many numbers, one needs to understand their asymptotic behaviors, or "virtual end numbers".

**Definition 1.1.** The limit of a sequence \( \{a_n\}_{n=1}^{\infty} \), if existed, is defined as a finite number \( L \) such that \( a_n \) can be as close to \( L \) as one wants as long as \( n \) is larger than a chosen number. More precisely, for any small number \( \varepsilon > 0 \) (for instance, \( \varepsilon = 0.001 \)), one can choose a very large number \( N(\varepsilon) \) (this number becomes extremely large when \( \varepsilon \) is very very small) satisfying

\[
|a_n - L| < \varepsilon \quad \text{for any } n > N(\varepsilon).
\]

This definition is consistent with the limit definition for functions. In particular, if \( a_n = f(n) \) is a sequence defined by a function \( y = f(x) \), then

\[
\lim_{n \to \infty} a_n = \lim_{x \to \infty} f(x) = L.
\]
Example 1.5. Sequence limits. (a)

\[
\lim_{n \to \infty} \frac{\ln n}{n} = \lim_{x \to \infty} \frac{\ln x}{x} \quad \text{L'Hospital's rule} \\
= \lim_{x \to \infty} \frac{(\ln x)'}{x'} = \lim_{x \to \infty} \frac{1}{x} = 0
\]

(b)

\[
\lim_{n \to \infty} \frac{5n^2 + 2n + 2}{n^2 + 3} = \lim_{n \to \infty} \frac{(5n^2 + 2n + 2) / n^2}{(n^2 + 3) / n^2} = \lim_{n \to \infty} \frac{5 \frac{2}{n} + \frac{2}{n^2}}{1 + \frac{3}{n^2}} = 5
\]

When we deal with limits of sequences, in general, we may apply "leading term principle":

\[
\lim_{n \to \infty} \frac{P(n)}{Q(n)} = \lim_{n \to \infty} \frac{\text{leading term of } P(n)}{\text{leading term of } Q(n)}
\]

For instance,

\[
\lim_{n \to \infty} \frac{5n^2 + 2n + 2}{n^2 + 3} = \lim_{n \to \infty} \frac{\text{leading term of } (5n^2 + 2n + 2)}{\text{leading term of } (n^2 + 3)}
\]

\[
= \lim_{n \to \infty} \frac{5n^2}{n^2} = 5
\]

\[
\lim_{n \to \infty} \frac{3n^5 + 2n^3 + 2n^2 - 5}{n^6 + 3n^5 + 6} = \lim_{n \to \infty} \frac{3n^5}{n^6} = \lim_{n \to \infty} \frac{3}{n} = 0
\]

\[
\lim_{n \to \infty} \frac{n^4 + 3n^3 - 2n^2 + 5}{n^2 + 3n - 6} = \lim_{n \to \infty} \frac{n^4}{n^2} = \lim_{n \to \infty} n^2 = \infty
\]

This leading term principle can also be applied "locally" at various levels. For instance,

\[
\lim_{n \to \infty} \frac{\sqrt{n^4 + 3n^3 - 2n^2 + 5}}{n^2 + 3n - 6} = \lim_{n \to \infty} \frac{\sqrt{n^4}}{n^2} = 1.
\]

Example 1.6. Geometric sequence \( \{r^n\}_{n=0}^\infty \). For instance,

\[
r = 2 : \quad 1, 2, 2^2, 2^3, 2^4, \ldots \to \infty
\]

\[
r = \frac{1}{2} : \quad 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots \to 0.
\]
In general,
\[
\lim_{n \to \infty} r^n = \begin{cases} 
0, & \text{if } |r| < 1 \\
\infty, & \text{if } r > 1 
\end{cases}
\]

**Example 1.7.** Alternating sequences.

\[
a_n = (-1)^n \frac{1}{n} : -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots \rightarrow 0
\]

\[
b_n = (-1)^n : -1, 1, -1, 1, -1, 1, \ldots \quad \text{It has no limit.}
\]

**Definition 1.2.** A sequence \( \{a_n\} \) is called **increasing** if

\[
a_n \leq a_{n+1}.
\]

If \( a_n \) can be represented by a function \( f(n) \), then sequence \( \{a_n\} \) is increasing means the underline function \( f(x) \) is increasing. A sequence \( \{a_n\} \) is called **decreasing** if

\[
a_n \geq a_{n+1}.
\]

A sequence \( \{a_n\} \) is called monotonic if it is either increasing or decreasing.

**Definition 1.3.** A sequence \( \{a_n\} \) is called **bounded above** if there is a finite number \( M \) such that

\[
a_n \leq M \quad \text{for all } n.
\]

The following sequence is bounded above:
The following is not bounded above:

**Definition 1.4.** A sequence \( \{a_n\} \) is called **bounded below** if there is a finite number \( m \) such that

\[ m \leq a_n \text{ for all } n. \]

A sequence \( \{a_n\} \) is called **bounded** if it is both bounded above and bounded below.

**Theorem.** (a) Any bounded monotonic sequence is convergent.
(b) Squeeze Theorem. If

\[ b_n \leq a_n \leq c_n \]

\[ \lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n = L. \]

Then

\[ \lim_{n \to \infty} a_n = L. \]

(c) Since

\[ -|a_n| \leq a_n \leq |a_n|, \]

\[ \lim_{n \to \infty} a_n = 0 \quad \text{if and only if} \quad \lim_{n \to \infty} |a_n| = 0 \]

**Example 1.8.** Consider the sequence:

\[ a_1 = 2, \quad a_{n+1} = \frac{a_n + 6}{2}. \]

We show by induction that

\[ 0 \leq a_n \leq 6 \quad \text{(bounded)} \]

\[ a_n \leq a_{n+1} \quad \text{(increasing)}. \]

When \( n = 1 \), both statements above are true, since \( a_2 = (2 + 6)/2 = 4 \).

Suppose they are true for \( n \leq k \), i.e.,

\[ 0 \leq a_n \leq 6 \quad \text{for} \ n \leq k \]

\[ a_n \leq a_{n+1} \quad \text{for} \ n \leq k. \]
We need to show they hold for \( n = k + 1 \), i.e.,
\[
0 \leq a_{k+1} \leq 6 \tag{1}
\]
\[
a_{k+1} \leq a_{k+2}. \tag{2}
\]

To show the first state (1), we note that since (by assumption)
\[
0 \leq a_k \leq 6,
\]
\[
a_{k+1} = \frac{a_k + 6}{2} \leq \frac{6 + 6}{2} = 6.
\]
The second statement (2) is verified by
\[
a_{k+1} = \frac{a_k + 6}{2} \leq \frac{a_{k+1} + 6}{2} \leq a_{k+2}.
\]
According to Theorem, the sequence is convergent, i.e., there exists a number \( L \) such that
\[
\lim_{n \to \infty} a_n = L.
\]

Now, if we apply limit to
\[
a_{n+1} = \frac{a_n + 6}{2} \implies \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{a_n + 6}{2}
\]
\[
\implies \lim_{n \to \infty} a_{n+1} = \frac{\lim_{n \to \infty} a_n + 6}{2} = L = \frac{L + 6}{2} = 2L = L + 6
\]
So, \( \lim_{n \to \infty} a_n = L = 6. \)

**Remark:** Changing a finite many items in a sequence doesn’t change the limit. For instance, consider two sequences
\[
\begin{align*}
(a) & \quad b_1, b_2, \ldots, b_{1000}, a_{1001}, a_{1002}, \ldots a_n, \ldots \\
(b) & \quad c_1, c_2, \ldots, c_{1000}, a_{1001}, a_{1002}, \ldots a_n, \ldots
\end{align*}
\]
Their first 1000 terms are different, but the rest terms are the same. But the change in the first 100 terms does not change their asymptotic behaviors. In fact, both sequences become identical when \( n > 1000 \). So these two sequences either both convergent or both divergent. If convergent, they have the same limit.

**Lab Assignment (due at the end of the lab session):** Lab #70

**Homework:**
1. List first 5 terms of the sequence defined by
   
   (a) \( a_n = (-1)^n \frac{n}{2n+1} \), \( n = 10, 11, 12, \ldots \)
   
   (b) \( b_0 = 2, \ b_{n+1} = \frac{5b_n - 2}{2}, \ n = 0, 1, 2, \ldots \)

2. Find a formula for the general term \( a_n \) of the following geometric sequence:
   
   (a) \( 1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots \)
   
   (b) \( 3.14, 3.1415, 3.141515, 3.14151515, \ldots \)
   
   (c) \( a_0 = 3, a_{n+1} = \frac{a_n}{3}, \text{ for } n = 0, 1, 2, \ldots \)

3. Determine whether the sequence converges or diverges. If converges, find the limit.
   
   (a) \( a_n = \frac{2n^3 + 1}{n^3 + 2n^2 + 3n + 5} \)
   
   (b) \( b_n = (-1)^n \frac{n}{n + 2} \)
   
   (c) \( c_n = (-1)^n \frac{n}{n^2 + 2} \)
   
   (d) \( d_n = \frac{(\ln n)^2}{n} \)

4. Find the first 40 terms of the sequence defined by
   
   \[
   a_{n+1} = \begin{cases} 
   \frac{a_n}{2}, & \text{if } a_n \text{ is an even number} \\ 
   3a_n + 1, & \text{if } a_n \text{ is an odd number} 
   \end{cases}
   \]

   and \( a_1 = 11 \). Do the same for \( a_1 = 25 \). Then make a conjecture for this type of series with any \( a_1 \).