Section 8 Inverse Trigonometric Functions

Inverse Sine Function

Recall that for every function \( y = f(x) \), one may define its INVERSE FUNCTION \( y = f^{-1}(x) \) as the unique solution of

\[
x = f(y).
\]

In other words, the inverse function \( y = f^{-1}(x) \) "undo" the function \( y = f(x) \):

\[
f(f^{-1}(x)) = x, \quad f^{-1}(f(x)) = x.
\]

We also know that even if a function \( y = f(x) \) is defined every, its inverse function \( y = f^{-1}(x) \) may not be defined everywhere. The sufficient condition that guarantees existence of inverse functions is called "horizontal line test": a horizontal line can intersect the graph of \( y = f(x) \) at most once. For trigonometric functions, for instance the graph of \( y = \sin x \) intersects horizontal \( y = 0.6 \) infinite many times:

Therefore, to define inverse function of \( y = \sin x \), we consider the restricted sine function
i.e., the restriction of \( y = \sin x \) on \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \). This function has the domain \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) and the range \([-1, 1]\); it captures all information of the sine function over all real numbers. Notice that the only difference between the restricted Sine function and the (unrestricted) Sine function is their domains:

\[
\text{restricted } \quad y = \sin x \text{ defined only for } x \text{ on } \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \\
y = \sin x \text{ defined for all real numbers } x
\]

**Definition:** The restricted Sine function defined on \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) is invertible, and its inverse function is denoted as

\[ y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x. \]

The domain of \( y = \arcsin x \) is \([-1, 1]\), and the range \( y = \arcsin x \) is \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \).

The graph of \( y = \arcsin x \) is the reflection of the graph for the restricted Sine function about \( y = x \):

![Graph](image)

**Example 1** Find the following inverse Sine function values:

(a) \( \arcsin \left( \frac{1}{2} \right) \)  (b) \( \arcsin \left( -\frac{\sqrt{3}}{2} \right) \)
Solution: (a) By the general definition of inverse functions, \( y = \arcsin \left( \frac{1}{2} \right) \) is the solution of the restricted Sine function for \( y \):

\[
\frac{1}{2} = \sin y
\]

The words "restricted Sine" means that

\[
-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.
\]

Therefore

\[
y = \frac{\pi}{6}, \quad \text{or} \quad \arcsin \left( \frac{1}{2} \right) = \frac{\pi}{6}
\]

(Notice that the notation "arcsin" comes from the fact that the length of the arc whose sine is 1/2 is \( \pi/6 \).)

(b) Similarly, we need to solve

\[
-\frac{\sqrt{3}}{2} = \sin y
\]

for

\[
-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.
\]

The solution is

\[
y = -\frac{\pi}{3}, \quad \text{or} \quad \arcsin \left( -\frac{\sqrt{3}}{2} \right) = -\frac{\pi}{3}
\]

Recall that by the definition of inverse function,

\[
f \left( f^{-1} (x) \right) = x \quad \text{for } x \text{ in } D \left( f^{-1} \right)
\]

\[
f^{-1} (f (x)) = x \quad \text{for } x \text{ in } D (f)
\]

Therefore,

\[
\sin (\arcsin x) = x \quad \text{for } x \text{ in } [-1, 1]
\]

\[
\arcsin (\sin x) = x \quad \text{for } x \text{ in } \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]
\]

Example 2 Find following values:
(a) \( \arcsin \left( \sin \left( \frac{\pi}{4} \right) \right) = \sin^{-1} \left( \sin \left( \frac{\pi}{4} \right) \right) = \frac{\pi}{4} \)

(b) \( \arcsin \left( \sin \left( \frac{3\pi}{4} \right) \right) = \sin^{-1} \left( \sin \left( \frac{3\pi}{4} \right) \right) = \frac{3\pi}{4} \)

(c) \( \arcsin \left( \sin \left( \frac{7\pi}{6} \right) \right) = \sin^{-1} \left( \sin \left( \frac{7\pi}{6} \right) \right) = \frac{\pi}{4} \)

**Solution:** (a) According to the formula above, since \( x = \frac{\pi}{4} \) is in \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \)

\[ \arcsin \left( \sin \left( \frac{\pi}{4} \right) \right) = \frac{\pi}{4} \]

(b) This time, since \( x = \frac{3\pi}{4} \) is **not** in \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \), the above formula that would have lead to

\[ \arcsin \left( \sin \left( \frac{3\pi}{4} \right) \right) = \frac{3\pi}{4} \] doesn’t apply.

We thus solve this problem, we need to find an angle \( x \) in \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) such that

\[ \sin x = \sin \left( \frac{3\pi}{4} \right). \]

This can be easily done as follows:

\[ \sin \left( \frac{3\pi}{4} \right) = \sin \left( \pi - \frac{\pi}{4} \right) = \sin \frac{\pi}{4}. \]

Thus

\[ \arcsin \left( \sin \left( \frac{3\pi}{4} \right) \right) = \arcsin \left( \sin \left( \frac{\pi}{4} \right) \right) = \frac{\pi}{4}. \]

(c) This time again \( \frac{7\pi}{6} \) is outside of the domain of the restricted sine function \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \). according to the method in (b),

\[ \sin \left( \frac{7\pi}{6} \right) = \sin \left( \pi + \frac{\pi}{6} \right) = -\sin \frac{\pi}{6} = \sin \left( -\frac{\pi}{6} \right). \]

Now since \( -\frac{\pi}{6} \) is in \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right], \)

\[ \arcsin \left( \sin \left( \frac{7\pi}{6} \right) \right) = \arcsin \left( \sin \left( -\frac{\pi}{6} \right) \right) = -\frac{\pi}{6}. \]
Example 3 Find the following values

(a) \( \sin \left( \arcsin \left( \frac{2}{3} \right) \right) = \sin \left( \sin^{-1} \left( \frac{2}{3} \right) \right) = \)

(b) \( \sin (\arcsin 2) = \sin \left( \sin^{-1} (2) \right) = \)

Solution: (a) Obviously, \( x = \frac{2}{3} \) is in the domain of \( \arcsin x \), \([-1, 1]\).

\[
\sin \left( \arcsin \left( \frac{2}{3} \right) \right) = \frac{2}{3}
\]

(b) \( x = 2 \) is not in \([-1, 1]\), \( \arcsin 2 \) is undefined. So \( \sin (\arcsin 2) \) is not defined.

Example 4 (a) What is \( \cos (\sin^{-1} x) \)? (b) What is \( \sin (2 \sin^{-1} x) \)?

Solution: (a) Set \( \theta = \sin^{-1} x \) which is defined for any \(-1 \leq x \leq 1\). This means

\[
\sin \theta = x \text{ and } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}
\]

Thus \( \cos \theta \geq 0 \), and

\[
\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - x^2}.
\]

(b) \[
\sin 2\theta = 2 \sin \theta \cos \theta = 2x\sqrt{1 - x^2}
\]

Inverse Cosine function
For \( y = \cos x \)

[Graph of the inverse cosine function]
we define the restricted Cosine function as

\[ y = \cos x \text{ for } x \text{ in } [0, \pi] \]

so that its graph is the piece in \([0, \pi]\)

Apparently, the restricted Cosine function passes the horizontal line test and thus is invertible. We call the inverse function of the restricted Cosine function inverse Cosine and is denoted by

\[ y = \cos^{-1} x \text{ or } y = \arccos x. \]

Analogous to the inverse sine function, there are some basic facts for \( y = \cos^{-1} x \):

\[ y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi] \]
\[
\cos(\cos^{-1} x) = x \text{ for } x \text{ in } [-1, 1] \\
\cos^{-1}(\cos x) = x \text{ for } x \text{ in } [0, \pi]
\]

**Example 5** Find the values:

(a) \(\cos^{-1}(0)\),  \(\cos^{-1}\left(-\frac{1}{2}\right)\)

*Solution:* (a) \(\theta = \cos^{-1}(0)\) is the angle in \([0, \pi]\) satisfying

\[
\cos \theta = 0 \implies \theta = \frac{\pi}{2}
\]

(b) \(\theta = \cos^{-1}\left(-\frac{1}{2}\right)\) is the solution of

\[
\cos \theta = -\frac{1}{2}, \text{ } \theta \text{ in } [0, \pi]
\]

Since \(\cos \frac{\pi}{3} = \frac{1}{2}\),

\[
\cos \frac{2\pi}{3} = \cos \left(\pi - \frac{\pi}{3}\right) = -\cos \frac{\pi}{3} = -\frac{1}{2}.
\]

So

\[
\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}
\]

**Example 6** compute

(a) \(\cos \left(\cos^{-1}\frac{2}{3}\right)\),  \(\arccos (\cos 4)\)

*Solution:* (a) Since \(x = \frac{2}{3}\) is in the domain of \(\arccos x\),

\[
\cos \left(\cos^{-1}\frac{2}{3}\right) = \frac{2}{3}
\]

(b) Note that \(x = 4 > \pi \approx 3.14\), which is beyond the domain of the restricted cosine. So the formula

\(\arccos (\cos x) = x\) doesn’t apply.
we need to find an angle \( \theta \) in \([0, \pi]\) such that \( \cos \theta = \cos 4 \). To this end

\[
\cos 4 = \cos (2\pi - 4), \quad \theta = 2\pi - 4 \text{ is in } [0, \pi]
\]

and

\[
\arccos (\cos 4) = \arccos (\cos \theta) = \theta = 2\pi - 4
\]

**Example 7** (a) *What is \( \sin (\cos^{-1} x) \)?* (b) *What is \( \sin^{-1} x + \cos^{-1} x \)\?*

**Solution:** (a) *Set \( \theta = \cos^{-1} x \) so \( \theta \) is in \([0, \pi]\) where \( \sin \theta > 0 \). Thus*

\[
\sin (\cos^{-1} x) = \sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - x^2}.
\]

(b) *Set \( \alpha = \sin^{-1} x \). Then \( x = \sin \alpha \). Draw a right triangle whose one angle is \( \alpha \), hypotenuse = 1, and the opposite side is \( a = x \). Then this side \( x \) is adjacent to the third angle in the triangle, i.e., \( \beta = \frac{\pi}{2} - \alpha \), and \( \cos \beta = x/1 \), or \( \beta = \cos^{-1} x \). We conclude*

\[
\sin^{-1} x + \cos^{-1} x = \alpha + \beta = \frac{\pi}{2}
\]

**Inverse function for Tangent Function**

Since the graph of \( y = \tan x \) is

![Graph of \( y = \tan x \)](image)

and the period is \( \pi \). So one entire period
passes the horizontal test and thus invertible. The inverse function of this restricted piece in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is denoted as

\[ y = \tan^{-1} x \quad \text{or} \quad y = \arctan x. \]

Some basic facts:

- $y = \tan^{-1} x$ has the domain $(-\infty, \infty)$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
  
\[ \tan^{-1} (\tan x) = x \quad \text{only for} \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \]

\[ \tan (\tan^{-1} x) = x \quad \text{for all} \quad x. \]

**Example 8** Find (a) $\tan^{-1} (-1)$, (b) $\tan (\tan^{-1} \sqrt{3})$
Solution: (a) Since \( \tan \left( -\frac{\pi}{4} \right) = -1 \), \( \tan^{-1}(-1) = -\frac{\pi}{4} \)

(b) \( \tan \left( \tan^{-1} \sqrt{5} \right) = \sqrt{5} \)

Example 9 (a) Given \( \tan \theta = x/2 \), \( 0 < \theta < \pi/2 \). Express \( \tan 2\theta \) as a function of \( x \).

(b) Simplify \( \sec (\tan^{-1} x) \)

Solution: (a) Using double-angle formula,

\[
\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2 \left( \frac{x}{2} \right)}{1 - \left( \frac{x}{2} \right)^2} = \frac{4x}{4 - x^2}
\]

(b) Set \( \theta = \tan^{-1} x \). \( x = \tan \theta \).

\[
\sec (\tan^{-1} x) = \sec x = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + x^2}
\]

Homework:
In Exercise 1-10, evaluate each of the quantities that is defined. If a quantity is undefined, explain it.

1. \( \sin^{-1} \left( \frac{\sqrt{3}}{2} \right) = \)

2. \( \cos^{-1} (-1) = \)

3. \( \tan^{-1} \sqrt{3} = \)

4. \( \sin \left( \sin^{-1} \left( -\frac{5}{4} \right) \right) = \)

5. \( \cos^{-1} \left( \cos \left( \frac{4\pi}{5} \right) \right) = \)

6. \( \tan^{-1} \left( \tan \left( -\frac{\pi}{5} \right) \right) = \)

7. \( \sin^{-1} \left( \sin \frac{4\pi}{3} \right) = \)
8. \( \cos\left(\cos^{-1}\left(\frac{3}{7}\right)\right) = \)

9. \( \sin\left(\sin^{-1}\left(\frac{3\pi}{2}\right)\right) = \)

10. \( \cos(\cos^{-1}2) = \)

   In Exercise 11-15, find and simplify the exact value of each quantity.

11. \( \tan\left(\sin^{-1}\left(\frac{4}{5}\right)\right) \)

12. \( \cos\left(\arcsin\left(\frac{4}{9}\right)\right) \)

13. \( \sin(\tan^{-1}2) \)

14. \( \sin\left(\cos^{-1}\left(\frac{1}{4}\right)\right) \)

15. \( \cos\left(\tan^{-1}\frac{4}{3}\right) \)

   Simplify the following expressions.

16. \( \tan(\cos^{-1}x) \)

17. \( \sin(\tan^{-1}x) \)

18. \( \cos(\sin^{-1}x + \cos^{-1}x) \)