Subjective Probability

Yan Liu

Department of Biomedical, Industrial & Human Factors Engineering
Wright State University

Subjective Interpretation of Probability

- Frequentist Interpretation of Probability
  - Long-term frequency of occurrence of an event
  - Requires one to conduct repeatable experiments
- What about the following events?
  - Core meltdown of a nuclear reactor?
  - You being alive at the age of 80?
  - Oregon beating Stanford in their 1970 football game? (unsure about the outcome although the event has happened)

Without doubt the above events are uncertain and we talk about their probabilities. These probabilities, however, are NOT relative frequencies of occurrences.

Subjective Interpretation: Probability = an individual's degree of belief in the occurrence of an event

Why Subjective Interpretation?

- Frequentist interpretation is not always appropriate
  - Some events cannot be repeated many times
    - e.g. Core meltdown of a nuclear power, your 80th birthday
  - The actual event has taken place, but you are uncertain about the result unless you have known the answer
    - e.g. The football game between Oregon and Stanford in 1970
- Subjective interpretation allows us to model and structure individualistic uncertainty through probability
  - Degree of belief that a particular event will occur can vary among different people and depend on the contexts

Measures of Uncertainty

- Probability
- Odds (For or Against)
  - Odds for: $\theta(A) = \frac{\Pr(A)}{\Pr(\overline{A})}$
  - Odds against: $\theta(\overline{A}) = \frac{\Pr(\overline{A})}{\Pr(A)}$
- Log Odds (For or Against)
  - Log odds for: $\log(\theta(A)) = \log(\Pr(A)) - \log(\Pr(\overline{A}))$
  - Log odds against: $\log(\theta(\overline{A})) = \log(\Pr(\overline{A})) - \log(\Pr(A))$
- Words
  - Fuzzy qualifiers
    - e.g. common, unusual, rare, etc.

Assessing Discrete Probabilities

- Direct Methods: Directly ask for probability assessment
  - The decision maker may not be able to give a direct answer
  - These methods do not work well if the probabilities in questions are small (such as in risk analysis)
- Indirect Methods: Formulate questions in the decision maker's domain of expertise and then extract probability assessment through probability modeling
  - Betting approach, Reference lottery

Assessing Discrete Probabilities

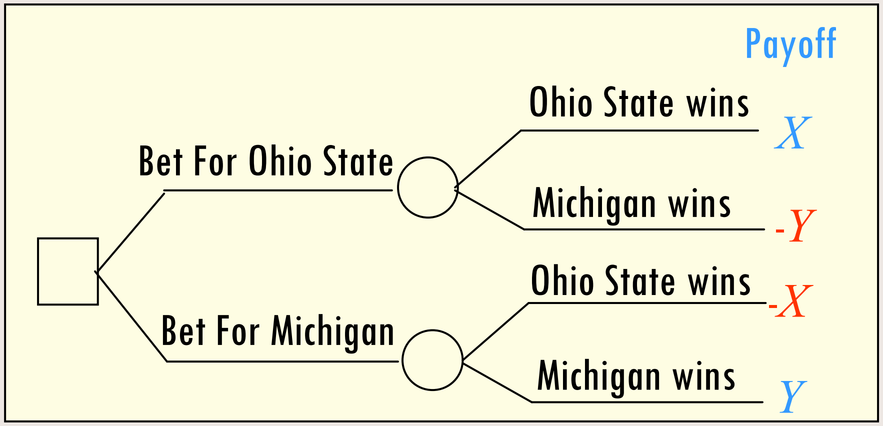
- Betting Approach
  - Find a specific amount to win or lose such that the decision maker is indifferent about which side of the bet to take

Football Example: Ohio State will play Michigan in football this year

➢Offer the decision maker to choose between the following bets:

Bet 1 – gain $$X$ if Ohio State wins and lose $$Y$ if Michigan wins

Bet 2 – Lose $$X$ if Ohio State wins and gain $$Y$ if Michigan wins



Assessing Discrete Probabilities

Football Example (Cont.)

Step 1: set X=100, Y=0 (or other numbers which make the preference obvious)

Step 2: set X=0, Y=100 (a consistency check; the decision should be switched)

Step 3: set X=100, Y=100 (the comparison is not obvious anymore)

Continue till the decision maker is indifferent between the two bets

Assumption: when the decision maker is indifferent between bets, the expected payoffs from the bets are the same

$X \cdot$Pr(Ohio State wins) $-Y \cdot$Pr(Michigan wins) $= -X \cdot$Pr(Ohio State wins)$+Y \cdot$Pr(Michigan wins)

=> Pr(Ohio State wins)=$Y/(X+Y)$

e.g. Point of indifference at X=70, Y=100, Pr(Ohio State wins)=0.59

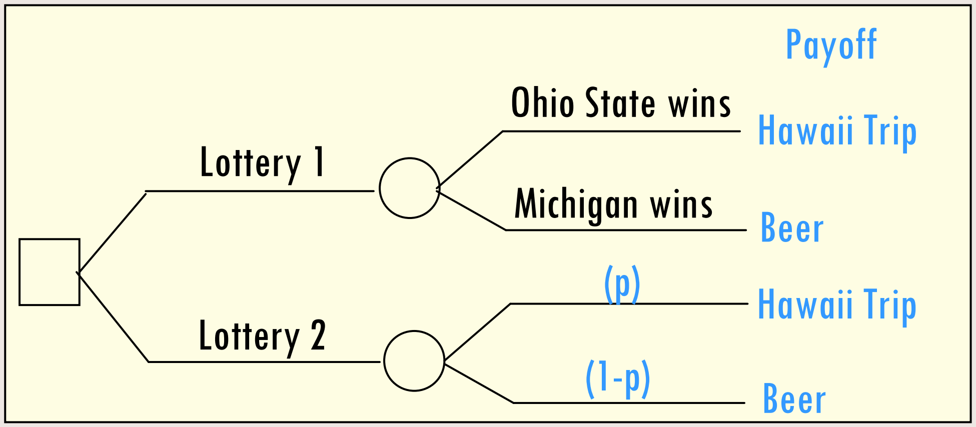
Assessing Discrete Probabilities

- Betting Approach (Cont.)
  - Is a straightforward approach
  - Many people do not like the idea of betting
  - Most people dislike the prospect of losing money (risk-averse)
  - Does not allow choosing any other bets to protect from losses

Assessing Discrete Probabilities

- Reference Lottery

Compare two lottery-like games, each resulting in a price (A or B, A>>B)



Lottery 2 is the reference lottery, in which a probability mechanism is specified (such as by drawing a colored ball from an urn or using "wheel of fortune")

The idea is to adjust the probability of winning in the reference lottery until the decision maker is indifferent between the two alternative lotteries.

Assessing Discrete Probabilities

- Reference Lottery (Cont.)

Step 1: Start with some p_1 and ask which lottery the decision maker prefers

Step 2: If lottery 1 is preferred, then choose p_2 higher than p_1; if the reference lottery is preferred, then choose p_2 less than p_1

Continue till the decision maker is indifferent between the two lotteries

Assumption: when the decision maker is indifferent between lotteries, Pr(Ohio State wins)=p

- Problems with Reference Lottery
  - Some people have difficulties making assessments in hypothetical games
  - Some people do not like the idea of a lottery-like game

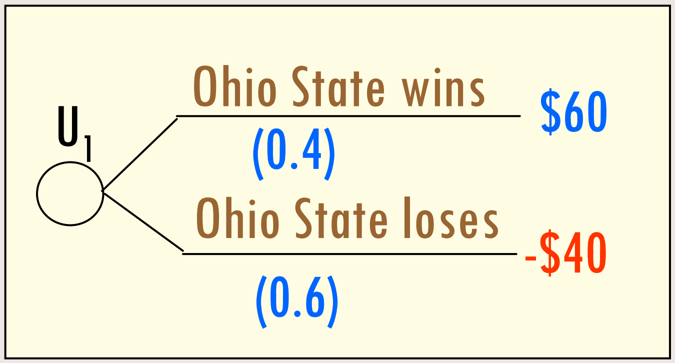
Coherence and The Dutch Book

- Subjective probabilities must follow the laws of probability. If they don't, the person assessing the probabilities is inconsistent
- Inconsistency may lead to a Dutch Book
- Dutch Book refers to a combinations of bets which, on the basis of deductive logic, can be shown to entail a sure loss

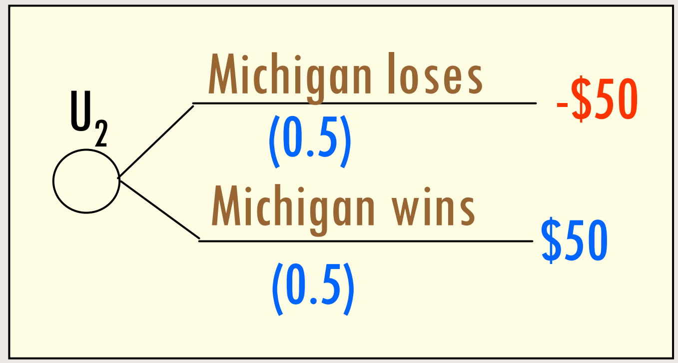
Football Example: Ohio State will play Michigan in football this year. Your friend says that Pr(Ohio State wins) = 40% and Pr(Michigan wins) = 50%. You note that the probabilities do not add up to 1, but your friend stubbornly refuses to change his estimates.
You think, "Great! Let us set up a series of bets."

Coherence and The Dutch Book

Football Example (Cont.)



Bet 1



Bet 2

$EMV(U_1) = EMV(U_2) = 0$, so the two bets should be considered as fair and your friend should be willing to engage in both

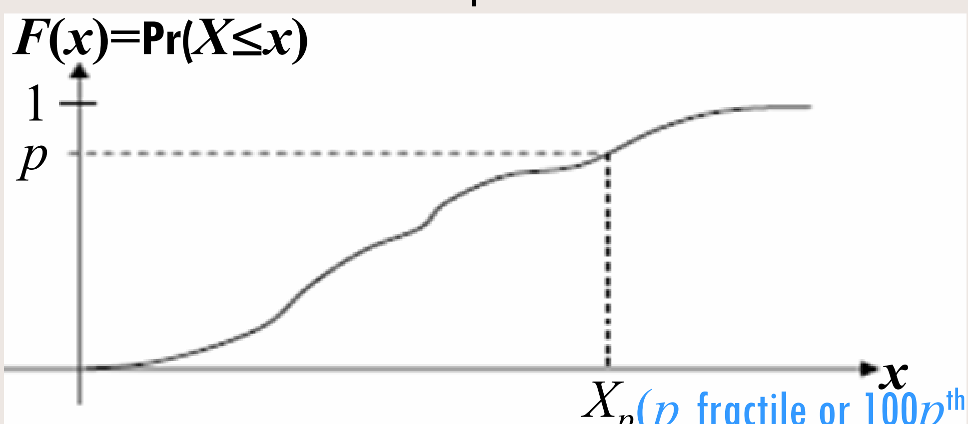
If Ohio State wins: you gain $60 in bet 1 but lose $50 in bet 2, so the net profit is $10

If Ohio State loses: you lose $40 in bet 1 but gain $50 in bet 2, so the net profit is $10

Whatever the outcome is, you will gain $10

Assessing Continuous Probabilities

- If the distribution of the uncertain event is known a priori, say normal distribution N(μ,σ), we can (more in Chapter 9)
  - Assess the distribution parameters directly
  - Assess distribution quantities and then solve for the parameters
- If the distribution is not known a priori, we can assess several cumulative probabilities and then use these probabilities to draft a CDF
  - Pick a few values and then assess their cumulative probabilities (fix x-axis)
  - Pick a few cumulative probabilities and then assess their corresponding x values (fix F(x)-axis)



X_p (p fractile or $100p^{th}$ percentile)

$X_{0.3}$ is 0.3 fractile or 30^{th} percentile

$X_{0.25}$ is lower quartile

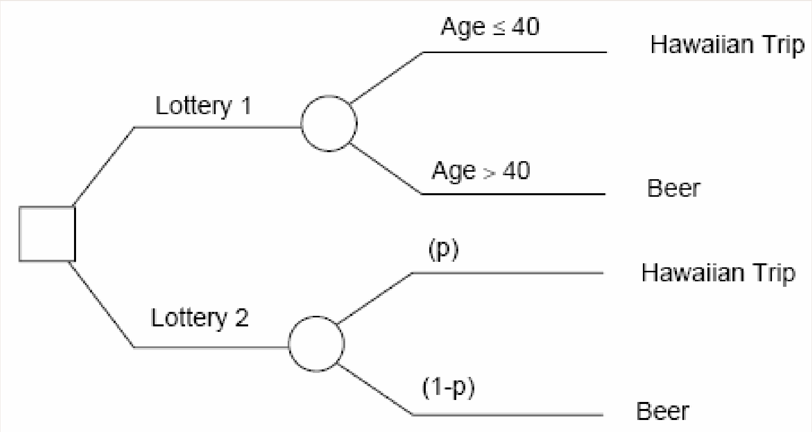
$X_{0.5}$ is median

$X_{0.75}$ is upper quartile

Assessing Continuous Probabilities

Example: An uncertain event is the current age, A, of a movie star

Method one: pick a few possible ages (say A=29, 40, 44, 50, and 65) and then assess their corresponding cumulative probabilities using the techniques for accessing discrete probabilities (such as the reference lottery approach)

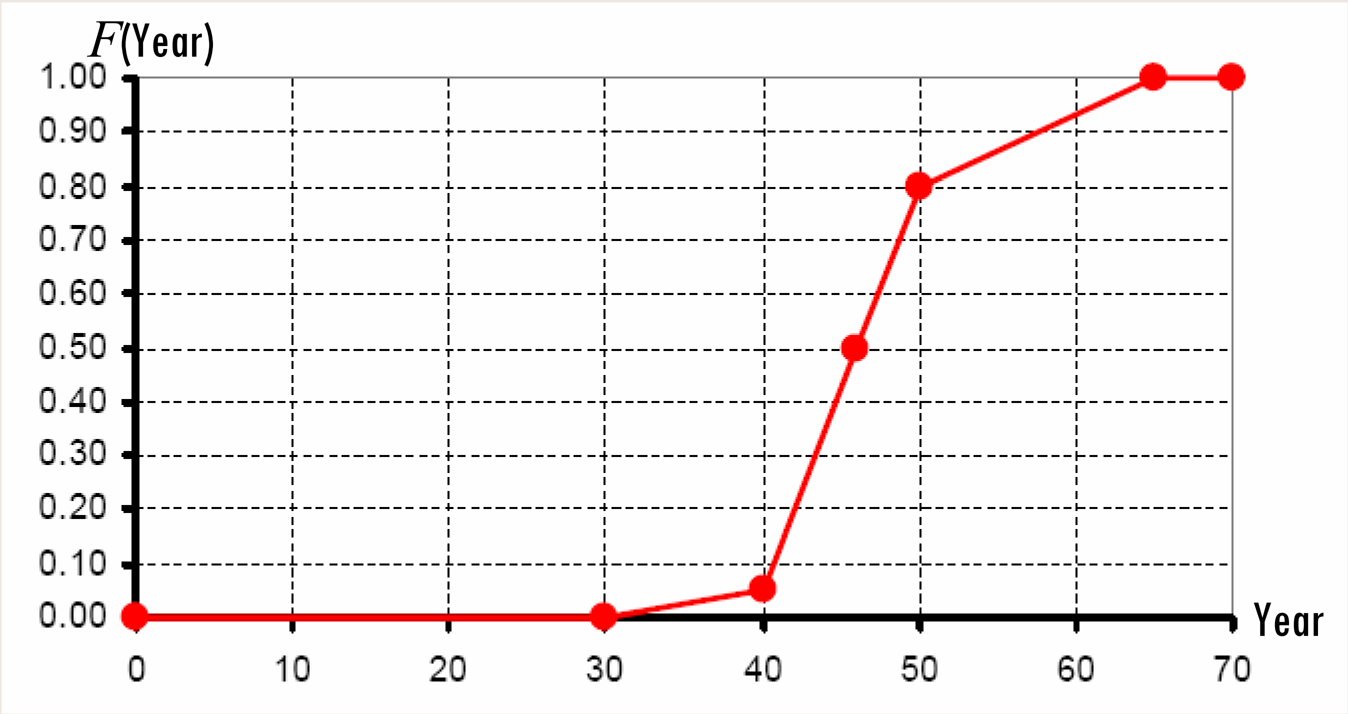


Reference Lottery for Assessing Pr (A≤40)

Suppose the following assessment were made:

$Pr(A\leq29)=0$, $Pr(A\leq40)=0.05$, $Pr(A\leq44)=0.5$, $Pr(A\leq50)=0.85$, $Pr(A\leq65)=1$

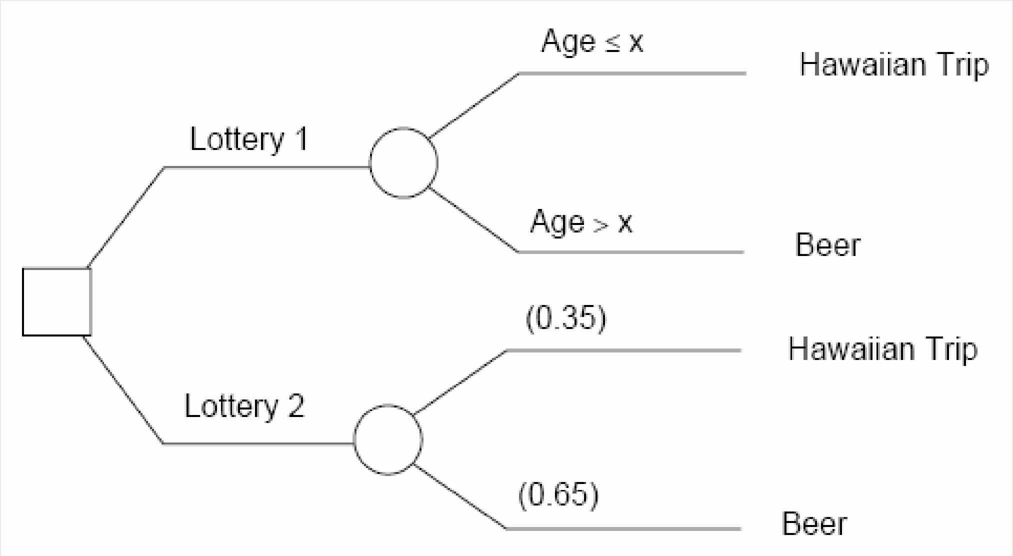
Assessing Continuous Probabilities



CDF of the Current Age of the Movie Star

Assessing Continuous Probabilities

Method two: pick a few cumulative probabilities and then assess their corresponding values using the reference lottery approach



Reference Lottery for Assessing the Value Whose Cumulative Prob. is 0.35 (the 35th percentile)

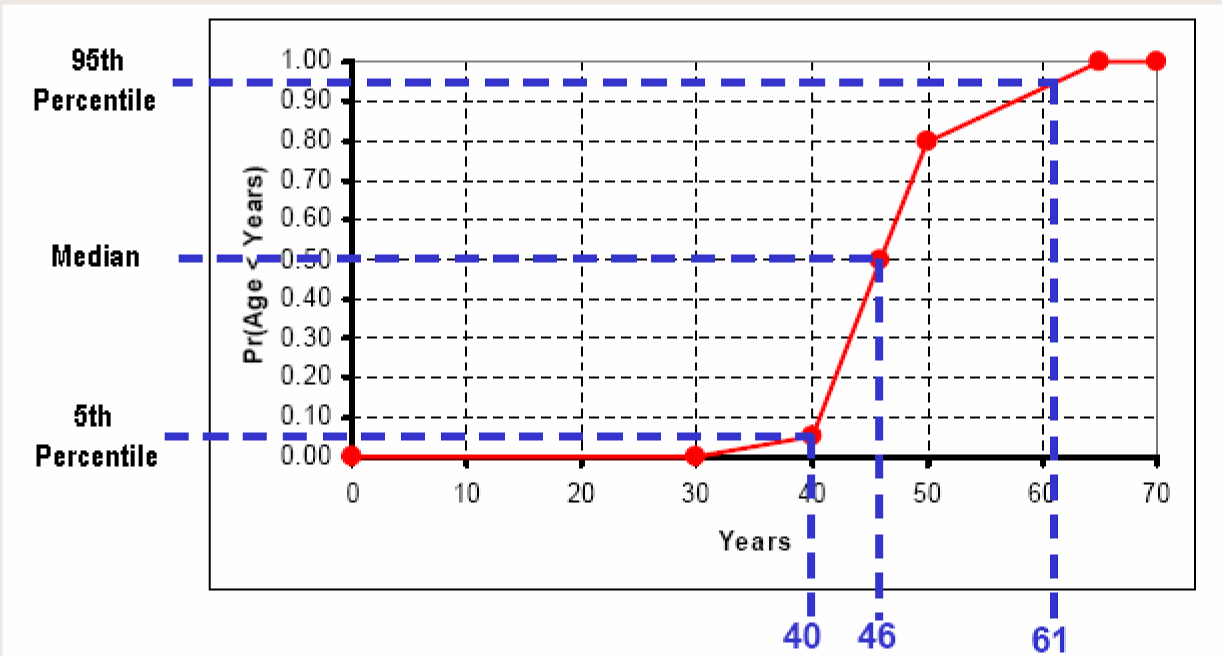
Typically, we assess extreme values (5th percentile and 95th percentile) , lower quartile (25th percentile), upper quartile (75th percentile) and median (50th percentile)

Using Continuous CDF in Decision Trees

- Advanced Approach: Monte Carlo Simulation (More in Chapter 11)
- Simple Approach: Approximate it with a discrete distribution
  - Use a few representative points in the distribution
  - Extended Pearson – Tukey method, Bracket median method
- Extended Pearson-Tukey method
  - Uses the median, 5th percentile, and 95th percentile to approximate the continuous chance node, and their probabilities are 0.63, 0.185, and 0.185, respectively
  - Works the best for approximating symmetric distributions

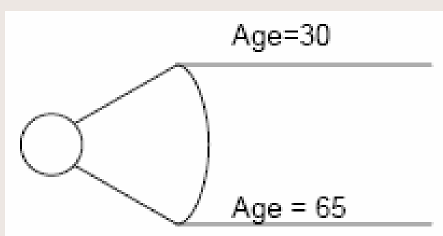
Using Continuous CDF in Decision Trees

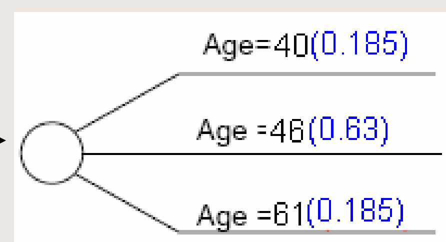
- Extended Pearson-Tukey method (Cont.)



CDF of the Age of the Movie Star

Continuous chance node





Discrete approximation

Using Continuous CDF in Decision Trees

- Bracket Median Method
  - The bracket median of interval [a,b] is a value m* between a and b such that Pr(a≤ X ≤ m*)=Pr(m*≤ X ≤ b)
  - Can approximate virtually any kind of distribution

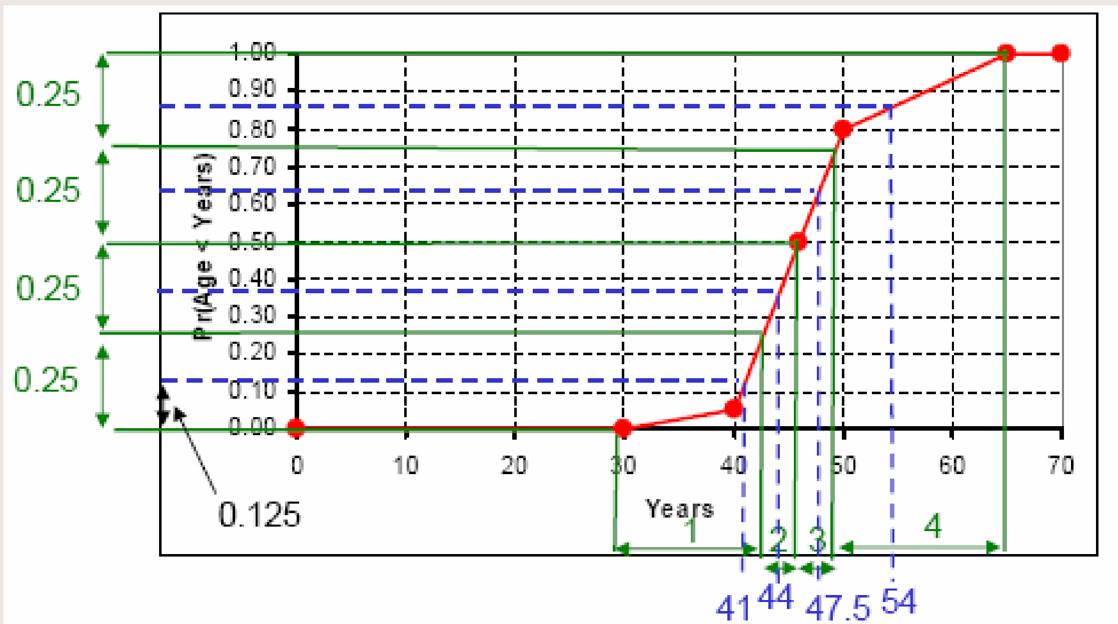
Step 1: Divide the entire range of probabilities into several equally likely intervals (typically three, four, or five intervals; the more intervals, the better the approximation yet more computations)

Step 2: Access the bracket median for each interval

Step 3: Assign equal probability to each bracket median (=100%/# of intervals)

Using Continuous CDF in Decision Trees

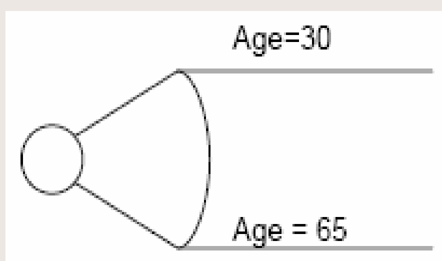
- Bracket Median Method (Cont.)

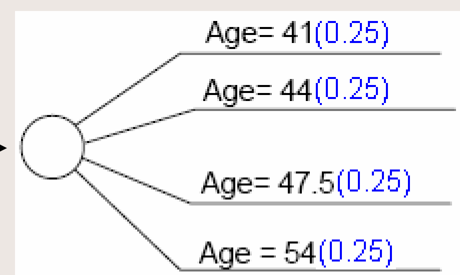


CDF of the Age of the Movie Star

4 intervals: [0,0.25],[0.25,0.5], [0.5,0.75], and [0.75,1]

Continuous chance node





Discrete approximation

Pitfalls: Heuristics and Biases

Thinking in terms of probability is NOT EASY!

Decision makers tend to use rather primitive cognitive techniques (heuristics) to make their probability assessment. These techniques are in general simple and intuitively appealing. However, they may result in a number of biases

- Representative Bias
  - Probability estimate is made on the basis of similarities within a group, while ignoring other relevant information, such as incidence/base rate (prior probability in Bayes theorem)
  - Insensitivity to the sample size
    - Draw conclusions from highly representative yet small samples although small samples are subject to large statistical errors (Law of Small Numbers)

Law of Large Numbers ?

Pitfalls: Heuristics and Biases

Example: X is the event that "a person is sloppily dressed". In your judgment, managers (M) are usually well dressed but computer scientists (C) are badly dressed. You think $\Pr(X\,|\,M)=0.1$, $\Pr(X\,|\,C)=0.8$.

Suppose at a conference with 90% attendance of managers and 10% attendance of computer scientists, you notice a person who is sloppily dressed. Do you think this person is more likely to be a manager or computer scientist?

$$\frac{\Pr(C|X)}{\Pr(M|X)} = \frac{\frac{\Pr(X|C)\Pr(C)}{\Pr(X)}}{\frac{\Pr(X|M)\Pr(M)}{\Pr(X)}} = \frac{\Pr(X|C)\Pr(C)}{\Pr(X|M)\Pr(M)} = \frac{0.8\cdot 0.1}{0.1\cdot 0.9} < 1$$

In conclusion, it is more likely this person is a manager than a computer scientist

Pitfalls: Heuristics and Biases

- Availability Bias
  - Probability estimate is made according to the ease with which one can retrieve similar events
    - Seeing a traffic accident can increase one's estimate of the chance of being in an accident
  - Illusory correlation: if two events are perceived as happening together frequently, this perception can lead to incorrect judgment regarding the strength of the relationship between the two events
    - You don't know very many people with visible tattoos, but you happen to observe that when fights break out, they seem to involve people with tattoos. You draw the conclusion that there must be an association between having tattoos and being prone to aggression

Pitfalls: Heuristics and Biases

- Anchoring-and-Adjusting Bias
  - One makes an initial assessment (anchor) and then make subsequent assessments by adjusting the anchor
    - People make sales forecasting by considering the sales figures for the most recent period and then adjusting those values based on new circumstances (usually insufficient adjustment)
  - Affects assessment of continuous probability distribution more often
- Motivational Bias
  - Incentives can lead people to report probabilities that do no entirely reflect their true beliefs
    - Weather forecasters tend to overstate the probability of rain, maybe because they would rather to have people be prepared for a bad weather and later pleasantly surprised by sunshine than expecting a good weather and being unpleasantly surprised

Decomposition and Probability Assessment

First break down a probability assessment into smaller and more manageable chunks using probability laws until a point at which the decision maker is comfortable with making assessment in a meaningful manner. Then, aggregate the detailed assessment using probability laws to obtain the answer to the original problem.

Steps to Decomposition

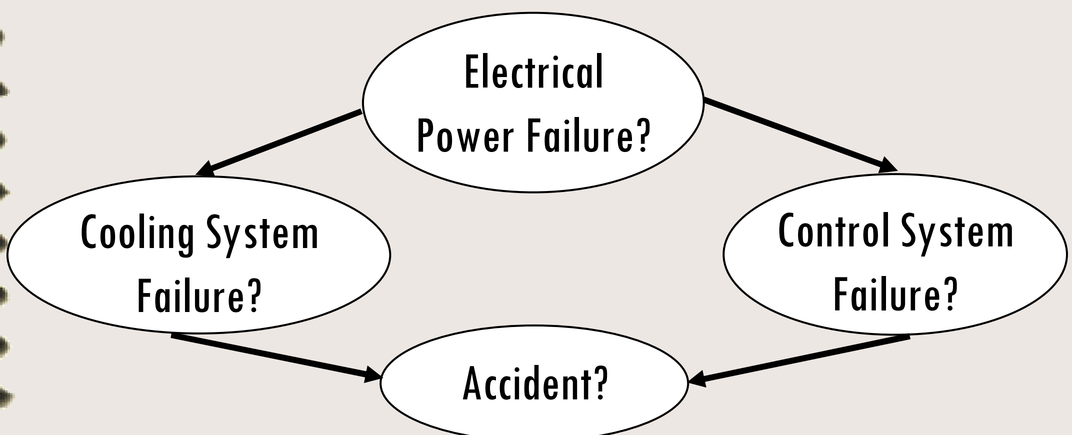
- Step 1: identify the conditioning events
- Step 2: Identify dependencies and sequences among events
- Step 3: Assess conditional probabilities
- Step 4: Aggregate detailed probability estimates to calculate the probability in question using probability rules

Decomposition and Probability Assessment

Radioactive Accident Example

Need to assess the probability of an accident resulting in the release of radioactive material into environment

- Failure of the cooling system or failure of the control system can lead to an accident
- Both the cooling system and control system depend on the electrical power system within the plant
- The cooling system and control system can be considered conditionally independent given the situation of the electrical power



A=Accident
L=Cooling system failure
N=Control system failure
E=Electrical system failure

Decomposition and Probability Assessment

Radioactive Accident Example

$$\Pr(A) = \Pr(A,(L,N)) + \Pr(A,(\bar{L},N)) + \Pr(A,(L,\bar{N})) + \Pr(A,(\bar{L},\bar{N}))$$

$$= \Pr(A \mid L,N)\,\underline{\Pr(L,N)} + \Pr(A \mid \bar{L},N)\,\underline{\Pr(\bar{L},N)} + \Pr(A \mid L,\bar{N})\,\underline{\Pr(L,\bar{N})} + \Pr(A \mid \bar{L},\bar{N})\,\underline{\Pr(\bar{L},\bar{N})}$$

$$\Pr(L,N) = \Pr(L,N,E) + \Pr(L,N,\bar{E}) = \underline{\Pr(L,N \mid E)}\,\Pr(E) + \underline{\Pr(L,N \mid \bar{E})}\,\Pr(\bar{E})$$

Because of the conditional independence between L and N given E

$$\Pr(L,N \mid E) = \Pr(L \mid E)\Pr(N \mid E) \quad \Pr(L,N \mid \bar{E}) = \Pr(L \mid \bar{E})\Pr(N \mid \bar{E})$$

Similarly, we can calculate $\Pr(\bar{L},N), \Pr(L,\bar{N}), \Pr(\bar{L},\bar{N})$

Therefore, in order to assess Pr(A), we need to assess the following conditional probabilities and probabilities:

$$\Pr(A \mid L,N), \Pr(A \mid \bar{L},N), \Pr(A \mid L,\bar{N}), \Pr(A \mid \bar{L},\bar{N}),$$

$$\Pr(L \mid E), \Pr(L \mid \bar{E}), \Pr(\bar{L} \mid E), \Pr(\bar{L} \mid \bar{E}), \Pr(N \mid E), \Pr(N \mid \bar{E}), \Pr(\bar{N} \mid E), \Pr(\bar{N} \mid \bar{E}),$$

$$\Pr(E), \Pr(\bar{E})$$

Finally, work backwards and substitute the appropriate values to calculate Pr(A)

Expert and Probability Assessment

- Experts are those knowledgeable about the subject matter
- Protocol for Expert Assessment
  - Identify the variables for which expert assessment is needed
    - Search for relevant scientific literature to establish scientific knowledge
  - Identify and recruit experts
    - In-house expertise, external recruitment
  - Motivate experts
    - Establish rapport and engender enthusiasm
  - Structure and decompose
    - Develop a general model (e.g. influence diagram) to reflect relationships among variables

Expert and Probability Assessment

- Protocols for Expert Assessment (Cont.)
  - Probability assessment training
    - Explain the principles of probability assessment and provide opportunities of practice prior to the formal elicitation task
  - Probability elicitation and verification
    - Make required probability assessments
    - Check for consistency
  - Aggregate Experts' Probability Distribution

Expert Assessment Principles

(Source: "Experts in Uncertainty" by Roger M. Cooke)

- Reproducibility
  - It must be possible for scientific peers to review and if necessary to reproduce all calculations (calculation models must be fully specified and data must be made available)
- Accountability
  - Source of expert judgment must be identified (their expertise levels and who they work for)
- Empirical Control
  - Expert probability assessment must in principle be susceptible to empirical control
- Neutrality
  - The method for combining/evaluating expert judgments should encourage experts to state true opinions
- Fairness
  - All experts are treated equally

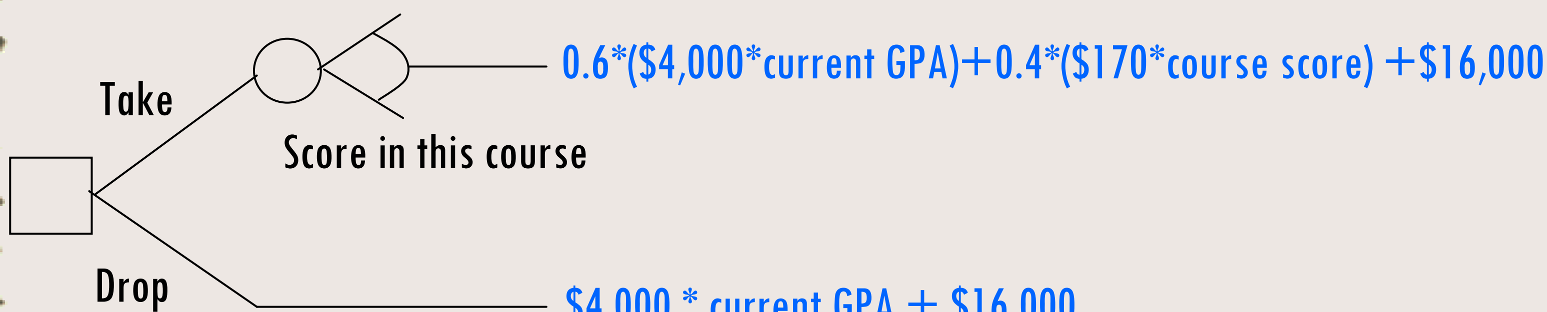
Exercise

Should you drop this course? Suppose you faced the following problem:

If you drop the course, the anticipated salary in your best job offer will depend on your current GPA: Anticipated salary | Drop = \$4,000 * current GPA + \$16,000

If you take the course, the anticipated salary in your best job offer will depend on both your current GPA and your score in this course:

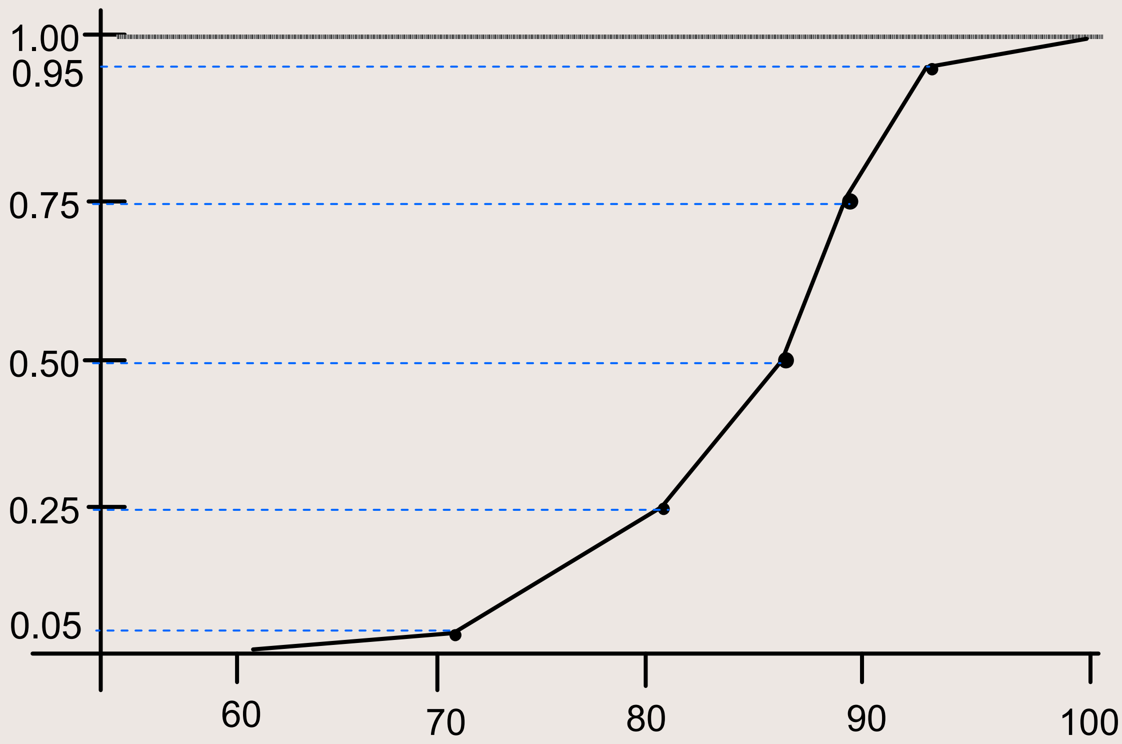
Anticipated salary | Take = 0.6*(\$4,000*current GPA)+0.4*(\$170*course score) +\$16,000



Let X = your score of this course, and assume: $X_{0.05}=71$, $X_{0.25}=81$, $X_{0.5}=87$, $X_{0.75}=89$, and $X_{0.95}=93$. Also assume your current GPA = 2.7

Exercise

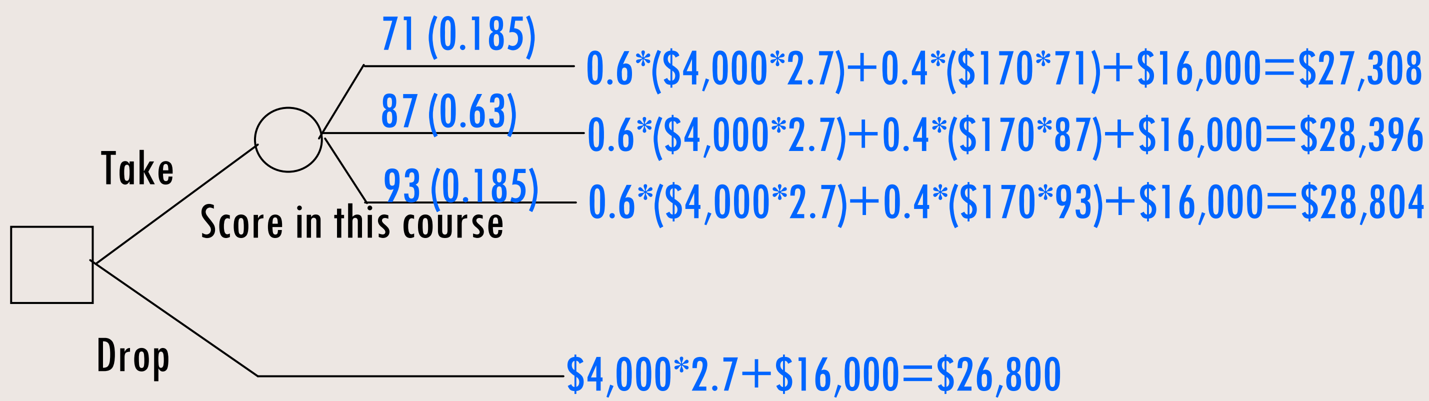
Based on the probability assessment, we can plot a CDF for X



Exercise

We can use either the Pearson-Tukey method or Bracket Median method to approximate the continuous chance node "score in the course" in the decision tree with discrete distributions

If the Pearson-Tukey method is used, since we have already estimate $X_{0.05}$ (=71), $X_{0.5}$(=87), and $X_{0.95}$(=93), we can plot the decision tree as follows



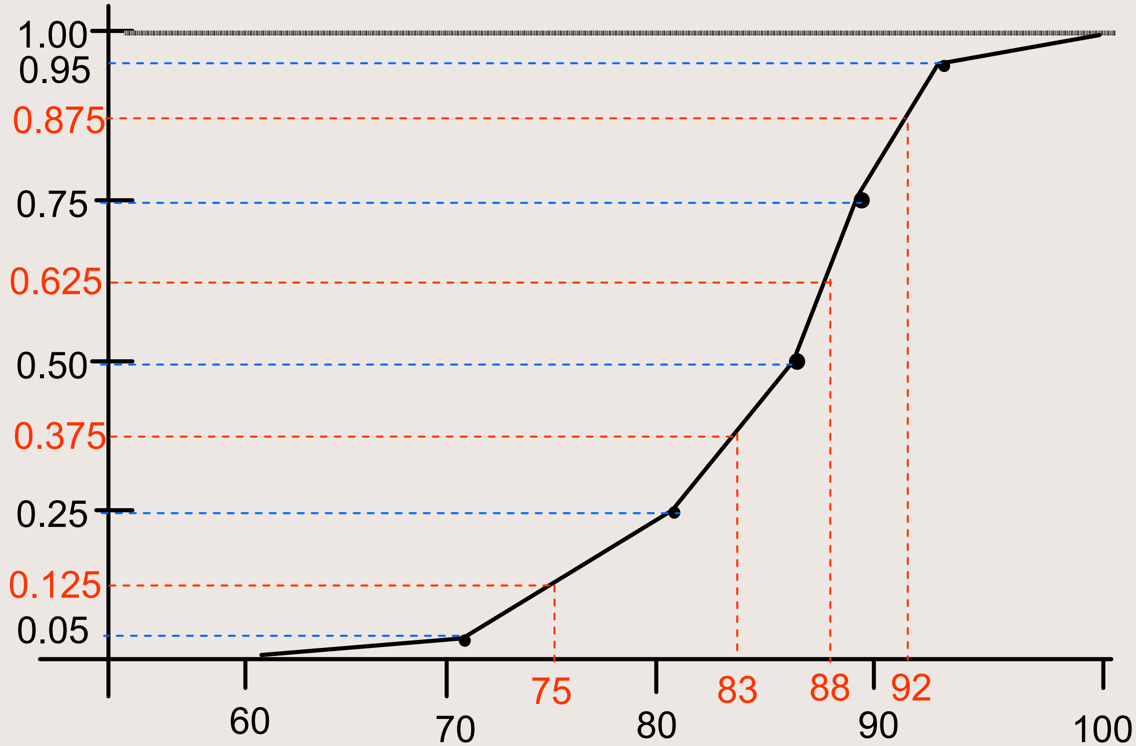
EMV(Take)=$27,308*0.185+$28,396*0.63+$28,804*0.185=$28,279,2

EMV(Take) > EMV(Dtrop), so you should take the course

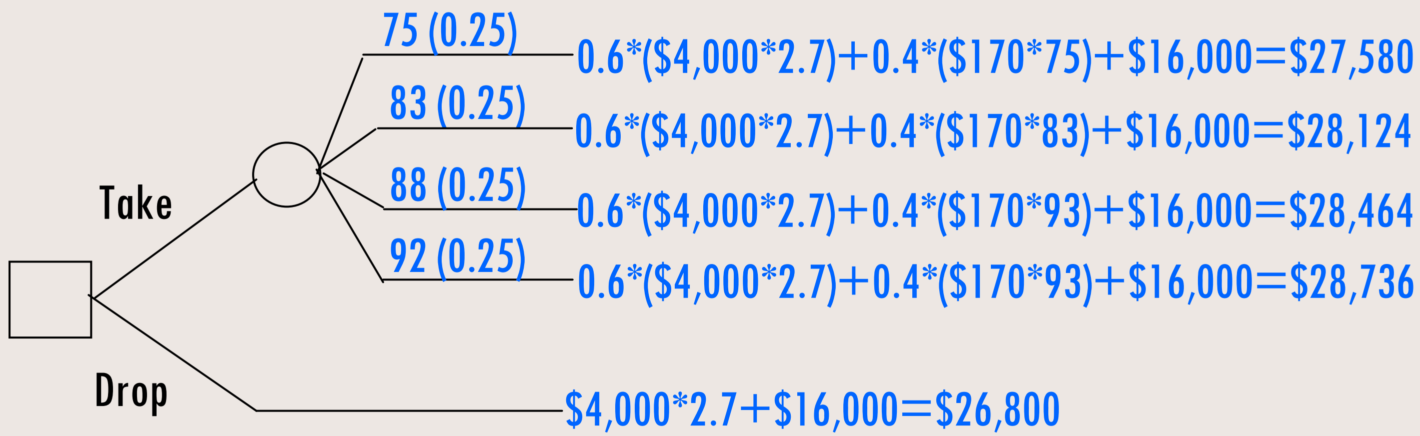
Exercise

If the Bracket Median method is used, we need to first divide the entire probability into intervals and then find the bracket medians of these intervals.

Suppose we divide the entire probability into four intervals: [0,0.25], [0.25,0.5], [0.5,0.75], and [0.75,1]



Exercise



EMV(Take)=$27,580*0.25+$28,124*0.25+$28,464*0.25+$28,736*0.25=$28,226

EMV(Take) > EMV(Dtrop), so you should take the course