On exact interval estimation for the odds ratio in subject-specific table

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We first show that the infimum coverage probability of a commonly used asymptotic confidence interval for the odds ratio is much less than the nominal level and then propose three exact intervals, which are also valid under mixed-effect models.

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1. Introduction

The odds ratio is often used to compare a treatment to a control for binary data. It is particularly important in the case-control study that uses a retrospective design to look into the past, since the difference and the ratio (relative risk) of two proportions cannot be applied. In a case-control study, the case group is typically matched with a control group according to certain profile to reduce possible confounding effects. However, it is more efficient to reduce such effects if each subject in the case group is matched to a subject in the control group. This results in a $2 \times 2 \times N$ subject-specific table which hopefully accounts for confounding effects, including the subject effect, as much as possible. Here $N$ is the number of subjects in the case group. We will derive three exact confidence intervals for the odds ratio that is defined in (3).

Consider $N$ matched pairs. A random vector $(Y_{i1}, Y_{i2})$ is observed from each pair $i = 1, \ldots, N$, where subscript 1 is for the control, and subscript 2 is for the case. Each $Y_{ij}$ assumes two values, 1 and 0. Assume a logit model for $Y_{i1}$ and $Y_{i2}$:

$$P(Y_{ij} = 1 | j) = \frac{e^{\alpha_i + \beta x_j}}{1 + e^{\alpha_i + \beta x_j}},$$

where $j = 1, 2$ and $i = 1, \ldots, N$, $x_1 = 0$ and $x_2 = 1$, and $\alpha_i$ and $\beta$ are unknown parameters. Among these $N + 1$ parameters, the $\alpha_i$'s are nuisance parameters accounting for the subject effect, and $\beta$ is the parameter of interest for the comparison between the case and control groups. Also, assume $(Y_{i1}, Y_{i2})$'s for $i = 1, \ldots, N$ are independent, and $Y_{i1}$ is independent of $Y_{i2}$. Different pairs may have different distributions indexed by $i$, due to possibly unequal $\alpha_i$'s. The joint probability mass function of all pairs is

$$p(\{y_{i1}, y_{i2}\}_{i=1}^N; \{\alpha_i\}_{i=1}^N, \beta) = \frac{e^{\sum_{i=1}^N \alpha_i(y_{i1} + y_{i2}) + \beta \sum_{i=1}^N y_{i2}}}{\prod_{i=1}^N (1 + e^{\alpha_i})[1 + e^{\alpha_i + \beta}]}.$$

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The subject-specific model is given by (1) and (2), and is described in Cox (1958), Rasch (1961) and Agresti (2002). We are interested in the comparison of the case $(j = 2)$ and the control $(j = 1)$ using their odds ratio,

$$r = \frac{P(Y_{i2} = 1|Y_{i1} = j = 2)}{P(Y_{i2} = 0|Y_{i1} = j = 2)} = \frac{P(Y_{i1} = 1|j = 1)}{P(Y_{i1} = 0|j = 1)} = e^{\beta}$$

(3)

since

$$r = \frac{P(j = 2|Y_{i2} = 1)}{P(j = 1|Y_{i2} = 1)} = \frac{P(j = 2|Y_{i1} = 0)}{P(j = 1|Y_{i1} = 0)}.$$  

(4)

The goal of this paper is to derive exact confidence intervals for $r$ for this purpose. The data structure is illustrated in the following example.

**Example 1.** Coulehan et al. (1986) discussed a case-control study of acute myocardial infarction (the MI case), where each of the 144 (= N) victims of MI was matched to one of 144 people free of heart disease (the MI control) according to age and gender. Each subject was asked whether he/she had ever been diagnosed as having diabetes, hypertension, obesity, etc. The diabetes effect on MI is of interest.

Let 1=diabetes and let 0=no diabetes. Each pair $(Y_{i1}, Y_{i2})$ has four possible outcomes, $(1,1),(0,1),(1,0)$ and $(0,0).$ A summary of their frequencies $n_{ui}$ is given in Table 1 for $u = 1, 0$ and $v = 1, 0$. Since $37 (= n_{01})$ is much larger than $16 (= n_{10}),$ there seems to be a positive association between MI and diabetes. We will confirm this in Section 4. The 144 pairs are independent, as are the two diabetes statuses $Y_{i1}$ and $Y_{i2}$ within each pair, since they are observed from different subjects.

For different pairs, the chance of having diabetes for each control subject may be different, leading to a large number (up to $N = 144$) of nuisance parameters $p_i$. However, the odds ratio $r$ in (3) does not depend on subjects. We will estimate $r$ to detect the association between diabetes and MI using the $n_{ui}$’s. The odds ratio in (4) is of more interest than the one in (3), since the former measures the diabetes effect on the status of MI, but the data set was collected to see the MI effect on the status of diabetes. Since the two odds ratios are identical, we can infer the odds ratio in (4). This makes the odds ratio important in the case-control study. $\Box$

In practice, an asymptotic interval (Agresti, 2002, p. 417) for $r$ is typically employed. However, an exact probability calculation in Section 3 shows that the interval has an incorrect confidence level, lower than claimed, even for large samples. Section 4 describes three exact optimal intervals for $r$. The interval construction is optimized to compute intervals given large samples. The proofs are postponed to Appendix A.

### 2. Preliminaries

It is clear from (2) that $S_1$, $S_2$, $S_3$, and $n_{11} + n_{01}$ are minimum sufficient statistics for the $\alpha$’s and $\beta$, where $S_i = Y_{i1} + Y_{i2}$ is the total of the $i$th pair, with three possible values: 0, 1 and 2. The presence of a large number of nuisance parameters (i.e., the $\alpha$’s) makes interval construction for $r$ challenging. Fortunately, we can eliminate the nuisance parameters by using two simpler statistics $n_{01}$ and $n^* \equiv n_{01} + n_{10}$.

The conditional distribution of $(Y_{i1}, Y_{i2})$ for given $S_i = s$ is

$$P(Y_{i1} = y_{i1}, Y_{i2} = y_{i2}|S_i = s) = \begin{cases} 1, & \text{if } S_i = 0, 2; \\ \frac{y_{i1}! y_{i2}!}{y_i!} (1 - p)^{y_i - y_{i1} - y_{i2}}, & \text{if } S_i = 1, \end{cases} $$

(5)

where $p = \frac{r}{1+r}$ (see Agresti, 2002, p. 416). This distribution only involves $r$ when $S_i = 1$. So, inferences about $r$ can be made based on the event $|S_i = 1|$. Here $n^*$ is the number of pairs $(Y_{i1}, Y_{i2})$ with $S_i = 1$. Let $\text{Bin}(n, p)$ denote a binomial distribution with $n$ trials and proportion $p$. The following lemma justifies the choice of the two statistics $n_{01}$ and $n^*$.

**Lemma 1.** The conditional distribution of $n_{01}$ for a given $n^* \in [1, N]$ is $\text{Bin}(n^*, \frac{r}{1+r})$.

The next result discusses the distribution of $n^*$, which is needed to estimate an upper bound of the infimum coverage probability (ICP) of intervals based on $(n_{01}, n^*)$. 

<table>
<thead>
<tr>
<th>MI controls</th>
<th>MI cases</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diabetes(1)</td>
<td>9(= n_{11})</td>
<td>16(= n_{10})</td>
</tr>
<tr>
<td>No Diabetes(0)</td>
<td>37(= n_{01})</td>
<td>82(= n_{00})</td>
</tr>
<tr>
<td>Total</td>
<td>46</td>
<td>98</td>
</tr>
</tbody>
</table>
Lemma 2. For a fixed vector \((\alpha_1, \ldots, \alpha_N)\), let \(\alpha_1^*, \ldots, \alpha_m^*\) denote all distinct \(\alpha_i\)'s, and let \(N_j\) be the number of \(\alpha_i\)'s which are equal to \(\alpha_j^*\) for \(j \in [1, m]\). Then \(N = \sum_{j=1}^{m} N_j\) and \(n^* = \sum_{j=1}^{m} Z_j\), where \(Z_j\) are independent and each \(Z_j \sim \text{Bino}(N_j, p_j)\) for

\[
p_j = \frac{q_j(1-q_j)(1+r)}{1+(r-1)q_j} \quad \text{and} \quad q_j = \frac{e^{s_j}}{1+e^{s_j}} \in [0, 1].
\]

Therefore, the distribution of \(n^*\) depends on the parameters \(\{N_j, q_j\}_{j=1}^{m}\). In particular, if the \(\alpha_i\)'s are all equal to \(\alpha_1^*\) (i.e., \(m_0 = 1\)), then \(n^* \sim \text{Bino}(N, p_1)\).

3. Evaluating the ICP of an approximate interval

Agresti (2002, p. 417) discussed an asymptotic \(1 - \alpha\) interval for \(r = e^6\),

\[
C_A(n_01, n^*) = \left[ \frac{n_01}{(n^* - n_01)A}, \frac{n_01A}{n^* - n_01} \right], \quad \text{for} \ A = \exp\left( \frac{Z}{\sqrt{n_01 + 1/n^* - n_01}} \right),
\]

which follows the delta method. However, \(C_A\) has two drawbacks. (i) \(C_A\) is not well defined over the entire sample space due to a possible zero value of \(n_01\) and/or \(n^* - n_01\). When \(n^* - n_01 = 0\), the interval is equal to \([\infty, \infty]\), and the lower limit of the interval, which by intuition should be a positive number, cannot be defined. (ii) There is no guarantee that the confidence level of this interval, which is defined to be the ICP over all possible values of \(\alpha_i\)'s and \(\beta\), is truly equal or close to \(1 - \alpha\), even though the interval claims so. As pointed out by Agresti (2002), Wang and Zhang (2014), and others, the \(1 - \alpha\) Wald interval, the simplest asymptotic interval for a proportion, always has a zero confidence level for any sample size. Both the Wald interval and (7) use asymptotic normality to establish the claimed confidence level of \(1 - \alpha\). It is hard to link a failure of the former with a success of the latter. Even worse in the current case, since (i) the coverage probability function of any interval for \(r\) is function of \(N\) \(\alpha_1\)'s \((N = 144\) in Example 1) and \(\beta\) and (ii) the interval (7) is not well defined, it is difficult to evaluate its coverage probability function even at a single parameter configuration \((\{\alpha_i\}_{i=1}^{N}, \beta)\). Consequently, the reliability of applying such an interval is questionable.

The coverage probability function for \(C_A\) is a function of \((\{\alpha_i\}_{i=1}^{N}, \beta)\), or equivalently, a function of \((\{N_i, q_i\}_{j=1}^{m_0}, r)\) for \(m_0 \in [1, N]\). For example, \((\alpha_1, \ldots, \alpha_1, \beta)\) is equivalent to \((N_1, q_1, r)\) for \(m_0 = 1\). So, we use the same notation \(\text{Cover}_{C_A}\)

\[
\text{Cover}_{C_A}(\{\alpha_i\}_{i=1}^{N}, \beta) = \text{Cover}_{C_A}(\{N_i, q_i\}_{j=1}^{m_0}, r) \overset{\text{def}}{=} P(r \in C_A(n_01, n^*))
\]

The ICP of \(C_A\) is defined to be

\[
\text{ICP}(C_A) \overset{\text{def}}{=} \inf_{m_0 \in [1, N] \cap q_1, \ldots q_{m_0} \in [0, 1]} \text{Cover}_{C_A}(\{N_i, q_i\}_{j=1}^{m_0}, r)
\]

Here is an attempt to obtain an upper bound for \(\text{ICP}(C_A)\).

(a) Let \(C_A(n_01, n^*) = [0, \infty]\), the largest possible interval for \(r\), if \(n_01 = 0\) or \(n_01 = n^*\). We do so to make the ICP\((C_A)\) larger.

(b) Rewrite (8),

\[
\text{ICP}(C_A) = \inf_{r \geq 0} \left[ \inf_{\{N_i, q_i\}_{j=1}^{m_0}} \text{Cover}_{C_A}(\{N_i, q_i\}_{j=1}^{m_0}, r) \right] \overset{\text{def}}{=} \inf_{r \geq 0} \text{Cover}(r)
\]

Following Lemmas 1 and 2, we have

\[
\text{Cover}(r) = \inf_{\{N_i, q_i\}_{j=1}^{m_0}} E [P(r \in C_A(n_01, n^*)|n^*)]
\]

\[
= \inf_{\{N_i, q_i\}_{j=1}^{m_0}} \sum_{n^*=0}^{N} \left[ \sum_{n_01=0}^{m_0} \text{pmf}(n^*; \{N_i, q_i\}_{j=1}^{m_0}) \right] \sum_{n_01=0}^{m_0} \text{pmf}(n^*; \{N_i, q_i\}_{j=1}^{m_0})
\]

where \(\text{pmf}(n^*; \{N_i, q_i\}_{j=1}^{m_0})\) is the probability mass function of \(n^*\). \(\text{Cover}(r)\) requires one to find the infimum of a high-dimensional function of \(q_j\)'s (if \(m_0\) is large), which would be computationally challenging. When all \(\alpha_i\)'s are equal, then \(m_0 = 1\). Let

\[
\text{Cover}_1(r) = \inf_{N_1, q_1} \sum_{n^*=0}^{N} \left[ \sum_{n_01=0}^{m_0} \text{pmf}(n^*; \{N_i, q_i\}_{j=1}^{m_0}) \right] \text{pmf}(n^*; \{N_i, q_i\}_{j=1}^{m_0})
\]

Then, \(\text{Cover}(r) \leq \text{Cover}_1(r)\), and \(\text{ICP}(C_A)\) is bounded by \(\text{Cover}_1(r)\), which can be computed by finding the infimum of a one-dimensional function of \(q_1\). For example, \(\text{ICP}(C_A)\) is bounded by 0.8744(= \(\text{Cover}_1(1001)\)) and 0.8703, respectively, for \(N = 144\) (as in Example 1) and \(N = 1000\) when the claimed nominal level \(1 - \alpha\) is 0.95. These upper bounds might not be
sharp due to the strong assumption of equal $\alpha_i$’s, but they already indicate a much smaller true confidence level than the claimed level 0.95. Fig. 1 also shows a low coverage $\text{Cover}_1(r)$ in a larger range of $r$, even for a large sample size $N = 1000$.

One may argue that the assumption of constant $\alpha_i$’s is too strong. We now compute $\text{Cover}_C((N_j, q_j)_{j=1}^{m_0}, r)$ when $m_0 = 2$ and $N_1 = N/2$ (i.e., half of $\alpha_i$’s are equal to $\alpha_1^*$ and the rest of $\alpha_i$’s are equal to $\alpha_2^*$) for $N$ even. Similar to (10), let $\text{Cover}_2(r)$ be

$$\text{inf}_{(N_j, q_j)_{j=1}^{m_0}} \sum_{n^*=0}^{N} \sum_{n_01=0}^{n^*} I_C(n_01, n^*)(r) \text{bino}(n_01; n^*, p_1) \text{pmf}(n^*; (N_j, q_j)_{j=1}^{m_0}),$$

which involves finding the infimum of two-dimensional function of $q_1$ and $q_2$. Here, $n^*$ is the sum of two independent binomials $\text{Bino}(N_1, p_1)$ and $\text{Bino}(N_2, p_2)$, where $p_1$ and $p_2$ are given in Lemma 2. More precisely,

$$\text{pmf}(n^*; (N_j, q_j)_{j=1}^{m_0}) = \sum_{z_1=0}^{n^*} \text{bino}(z_1; N_1, p_1) \text{bino}(n^*-z_1; N_2, p_2).$$

Then $\text{Cover}_2(1001) = 0.8744$ for $N = 144$, and $\text{Cover}_2(7021) = 0.8703$ for $N = 1000$. Fig. 1 also includes $\text{Cover}_2(r)$. Clearly, the coverage of $C_A$ is below 0.95 for a large range of $r$, even for a large sample size $N = 1000$.

4. Three exact intervals for $r$

There are three kinds of exact intervals: lower one-sided, upper one-sided and two-sided intervals. The smallest one-sided confidence intervals for $r$ based on $(n_{01}, n^*)$ can be derived as follows.

**Lemma 3.** Two exact upper and lower $1 - \alpha$ confidence intervals for $r$ are

$$[0, U_r(n_{01}, n^*)] = \left\{ \begin{array}{ll}
0, & \text{if } n^* > 0; \\
1 - U_C(n_{01}, n^*), & \text{if } n^* = 0,
\end{array} \right.$$
and

\[
[L_r(n_0, n^*), +\infty) = \begin{cases} 
\left[ \frac{L_{CP}(n_0, n^*)}{1 - L_{CP}(n_0, n^*)}, +\infty \right], & \text{if } n^* > 0; \\
[0, +\infty), & \text{if } n^* = 0,
\end{cases}
\]  

(13)

where \([L_{CP}(X, n), 1]\) and \([0, U_{CP}(X, n)\] are the one-sided \(1 - \alpha\) Clopper–Pearson intervals for \(p\) based on a binomial observation \(X \sim \text{Bin}(n, p)\).

**Remark 1.** Any \(1 - \alpha\) interval \(C(n_0, n^*) = [L(n_0, n^*), U(n_0, n^*)]\) for \(r\) at \(n^* = 0\) must be equal to \([0, +\infty)\) to maintain its coverage probability no smaller than \(1 - \alpha\). This justifies our choice at \(n^* = 0\). \(\Box\)

**Remark 2.** \([L_{CP}(X, n), 1]\) is the smallest \(1 - \alpha\) one-sided interval for \(p\), and

\[
L_{CP}(x, n) = \begin{cases} 
x, & \text{if } x = 0; \\
\frac{xf_{2x,2(n-x+1),1-\alpha}}{n-x+1+xf_{2x,2(n-x+1),1-\alpha}}, & \text{otherwise},
\end{cases}
\]

(14)

where \(f_{u,v,c}\) denotes the upper \(c\)th percentile of an F-distribution with \(u\) and \(v\) degrees of freedom; and \(U_{CP}(X, n) = 1 - L_{CP}(n - X, n)\).

We now propose a two-sided \(1 - \alpha\) interval for \(r\). Wang (2014) derived an exact admissible \(1 - \alpha\) interval \(C_W(X, n) = [L_W(X, n), U_W(X, n)]\) for \(p\), where \(X \sim \text{Bin}(n, p)\). His interval is obtained by an iterative construction that uniformly improves (shrinks) the \(1 - \alpha\) Clopper and Pearson (1934) in two-sided interval in a predetermined order on each \(x\)-value of \(0, 1, \ldots, n\) while the ICP is always maintained no smaller than \(1 - \alpha\). The construction starts at \(x = \lfloor n/2 \rfloor\) and goes down to \(x = 0\), and the interval satisfies \(U_W(X, n) = 1 - L_W(n - X, n)\). This interval is admissible under the set inclusion criterion and has a better performance on interval length than the Blyth and Still (1983) interval. More importantly, an R-code is available to implement the interval.

For a given \(n^* = 1, \ldots, N\), we compute the \(1 - \alpha\) Wang interval (2014) \([L_W(n_0, n^*), U_W(n_0, n^*)]\) for \(p_r\) based on \(n_0\) due to Lemma 1; for \(n^* = 0\), we estimate \(p_r\) by \([0, 1]\). Therefore, an interval is defined on the entire sample space in terms of \((n_0, n^*)\):

\[S = \{(n_0, n^*) : 0 \leq n_0 \leq n^* \leq N\}.\]

Therefore,

\[P(p_r \in [L_W(n_0, n^*), U_W(n_0, n^*)] | n^*) \geq 1 - \alpha, \text{ for any } p_r \in [0, 1] \text{ and } n^* \geq 0.\]

(15)

Lastly, we convert it to an interval for \(r\) by equating \(r = p_r/(1 - p_r)\). The proposed two-sided \(1 - \alpha\) interval for \(r\) in this paper is as follows

\[C_r(n_0, n^*) = \begin{cases} 
\left[ \frac{L_W(n_0, n^*)}{1 - L_W(n_0, n^*)}, \frac{U_W(n_0, n^*)}{1 - U_W(n_0, n^*)} \right], & \text{if } n^* \in [1, N]; \\
[0, +\infty), & \text{if } n^* = 0.
\end{cases}
\]

(16)

**Theorem 1.** The interval (16) is an exact \(1 - \alpha\) interval for \(r\).

**Example 1 (continued).** Here \(n_0 = 37\) and \(n^* = 53\). The 95% interval \(C_\text{A}(37, 53)\) is equal to [1.2865, 4.1569]. In contrast, the 95% Wang interval (2014) \([C_W(37, 53), n^*]\) is \([0.5565, 0.8130]\), which is obtained by running an R-code available from the author. Then the 95% exact interval \(C_r\) for \(r\) is \([1.2551, 4.3468]\) following (16). It indicates a positive association between MI and diabetes. In order to have a relatively fair comparison between \(C_r\) and \(C_\text{A}\), the 87.44% interval \(C_r = [1.4503, 3.7963]\) is computed, which has 81.73% the length of the 95% interval \(C_\text{A} = [1.2865, 4.1569]\), whose true ICP is at most 0.8744. \(\Box\)

**Remark 3.** One can also apply a mixed-effects model to reduce the large number of nuisance parameters \(\alpha_i\). For example, assume \((1)\) and \((2)\) and \(\alpha_i \sim \text{iid}, \text{N}(\mu, \sigma^2)\) with unknown \(\mu\) and \(\sigma^2\). See Lindsay, Clogg, and Grego (1991) and Neuhaus, Kalbfleisch, and Hauck (1994). The intervals \((12), (13)\) and \((16)\) are still exact \(1 - \alpha\) intervals for \(r\) when the mixed-effects model is assumed.

5. Summary

As discussed in Section 3, asymptotic intervals may have a low coverage probability even with a large sample, so exact intervals are needed for reliable inferences. The presence of nuisance parameters typically makes it more complex to derive an exact confidence interval, especially when a pivotal quantity cannot be found and the number of nuisance parameters is large. When making inferences on the odds ratio, which plays the most important role in the case-control study, the
conditioning technique is employed to eliminate all nuisance parameters, yielding a one-dimensional statistic \( r_{01} \), whose distribution only depends on the odds ratio. Therefore, the interval construction is also one-dimensional, and exact optimal intervals are derived. Because these intervals are of level \( 1 - \alpha \) for each set of fixed \( \alpha_i \)'s, following the conditional probability argument, they have level \( 1 - \alpha \) as well when the \( \alpha_i \)'s are random. i.e., the intervals are also valid under mixed-effects models where the \( \alpha_i \)'s are unobserved random variables, and \( \beta \) is a unknown constant. Lastly, R-codes for implementing the proposed intervals in (12), (13) and (16) are available from the author.

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Appendix A. Proofs

Proof of Lemma 1. For a set of observed \((Y_{i1}, Y_{i2})\)'s, let \( i = [i \in [1, N] : S_i = 1] \). Then the number of elements in \( I \), denoted by \( |I| \), is equal to \( n^* \) and \( n_{01} = \sum_{i \in I} Y_{i2} \). The joint distribution of \((Y_{i1}, Y_{i2})\)'s for a given nonempty set \( I \) is

\[
\prod_{i \in I} p_{r}^{y_2(i)} (1 - p_r)^{1 - y_2} = p_{r}^{n_{01}} (1 - p_r)^{n^* - n_{01}}.
\]

Thus, the conditional probability \( P(n_{01} = u | I) \), for a nonnegative integer \( u \leq n^* \), is

\[
P(n_{01} = u | I) = \frac{n^!}{u!(n^* - u)!} p_r^{u} (1 - p_r)^{n^* - u},
\]

which depends on the set \( I \) through \( n^* \), and

\[
P(n_{01} = u | n^*) = \frac{\sum\{all \; sets \; I \; with \; |I| = n^*\} P(n_{01} = u | I) P(I)}{\sum\{all \; sets \; I \; with \; |I| = n^*\} P(I)} = P(n_{01} = u | I).
\]

This establishes the lemma. \( \Box \)

Proof of Lemma 2. It is clear that \( N = \sum_{j=1}^{m_o} N_j \). For each \( \alpha_j \) for \( j = 1, \ldots, m_o \), let \( Z_j \) be the number of those pairs \((Y_{i1}, Y_{i2})\) that are equal to \((1,0)\) or \((0,1)\) and have \( \alpha_i = \alpha_j^* \). Then \( Z_j \sim \text{Bino}(N_j, p_j) \) with

\[
p_j = P(Y_{i1}, Y_{i2} = (1, 0)) + P(Y_{i1}, Y_{i2} = (0, 1)) = \frac{\alpha_j^*}{1 + \alpha_j^* + \beta} + \frac{1}{1 + \alpha_j^* + \beta} = \frac{q_j(1 - q_j)(1 + r)}{1 + (r - 1)q_j}.
\]

Therefore, \( n^* = \sum_{j=1}^{m_o} Z_j \) is the sum of independent binomials as claimed. \( \Box \)

Proof of Remark 1. Suppose the claim is not true. Then \([L(n_{01}, 0), U(n_{01}, 0)] = [A, B]\), where either \( A > 0 \) or \( B < +\infty \). Pick \( \beta_0 \notin [A, B] \). Thus,

\[
\text{Cover}_C(\{\alpha_i\}_{i=1}^{N}) \leq 1 - P(n^* = 0) \leq 1 - P(Y_{i1}, Y_{i2} = (1, 1), i = 1, \ldots, N) = 1 - \prod_{i=1}^{N} \frac{\alpha_i}{1 + \alpha_i + \beta_0} \]

Let all \( \alpha_i \)'s go to \( +\infty \). Then, the coverage probability converges to zero, contradicting the fact that the interval is of level \( 1 - \alpha \). Hence, \([L(n_{01}, 0), U(n_{01}, 0)] = [0, +\infty) \). \( \Box \)

Proof of Theorem 1. The coverage probability of the interval (16) satisfies

\[
\text{Cover}_C(\{\alpha_i\}_{i=1}^{N}, \beta) \overset{\text{def}}{=} P(r \in [L_r(n_{01}, n^*), U_r(n_{01}, n^*)])
\]

\[
= P(p_r \in [L_W(n_{01}, n^*), U_W(n_{01}, n^*)], n^* \in [1, N]) + P(p_r \in [0, 1], n^* = 0)
\]

\[
= \sum_{i=1}^{N} P(p_r \in [L_W(n_{01}, i), U_W(n_{01}, i)], n^* = i) P(n^* = i) + P(n^* = 0)
\]

\[
\geq (1 - \alpha) P(n^* = i) + P(n^* = 0) \geq 1 - \alpha. \Box
\]

Appendix B. Supplementary data

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.spl.2017.06.016.
References