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Testing against second-order stochastic dominance of multiple distributions

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Second-order stochastic dominance plays an important role in reliability and various branches of economics such as finance and decision-making under risk, and statistical testing for the stochastic dominance is often useful in practice. In this paper, we present a test of stochastic equality under the constraint of second-order stochastic dominance based on the theory of empirical processes. The asymptotic distribution of the test statistic is obtained, and a simple method to compute the critical value is derived. Simulation results and real data examples are presented to illustrate the proposed test method.

Keywords: Second-order stochastic dominance; asymptotic distribution; hypothesis testing; weak convergence.

Mathematics Subject Classification 2010: 62E15, 62H10

1. Introduction

The order between probability distributions plays an important role in many scientific areas including reliability analysis and economic research. Many types of orderings of varying degrees of strength for comparing univariate probability distributions have been discussed in the literature. Among them are likelihood ratio ordering, uniform stochastic ordering and the first- and second-order stochastic

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dominance, introduced in [2, 3, 8], respectively. The second-order stochastic dominance has been used in finance to develop a general framework for establishing a criterion for selecting one option over another. Other applications of it refer to, among others, [10, 11].

On the other hand, in many applications, we are faced with how to test some order between the univariate probability distributions. In this paper, we focus on the test of second-order stochastic dominance. Testing for second-order stochastic dominance of two distributions has been widely studied by many authors, see, for example, [4, 5, 9]. Although statistical inference to compare two populations under second-order stochastic dominance has a long and rich history, multiple comparisons have not been well studied. In this paper, we propose a test on the stochastic equality of multiple distributions against the stochastic monotonicity under second-order stochastic dominance.

In Sec. 2, we introduce the definition of second-order stochastic dominance and its equivalent expression. In Sec. 3, we firstly obtain consistent estimators of the integrated distributions functions, then give the asymptotic distribution of the estimators. In Sec. 4, we present a test for the stochastic equality of the multiple distributions based on the consistent estimators. In Secs. 5 and 6, we give simulation results and real examples to illustrate the performance of the proposed method. Some conclusion remarks are given in Sec. 7.

2. Second-Order Stochastic Dominance and Its Equivalent Expression

For the convenience of statement, we first recall the definition of second-order stochastic dominance.

Definition 1. Let X and Y be random variables with cumulative distribution functions (c.d.f.) F and G, respectively. We say that X dominates Y in the sense of second-order stochastic dominance, and denote by $X \geq_{\mathrm{SSD}} Y$ or $F \geq_{\mathrm{SSD}} G$, if $E(u(X)) \geq E(u(Y))$ for every nondecreasing and concave function u whenever the expectations exist and are finite.

The following lemma gives an equivalent statement of the second-order stochastic dominance.

Lemma 1. *If X and Y be independent random variables with corresponding c.d.f.'s F_1 and F_2, where $\lim_{x\to-\infty} xF_i(x) = 0$, $i = 1, 2$, then the following two statements are equivalent*:

(i) $F_1 \geq_{\mathrm{SSD}} F_2$.

(ii) $$\int_{-\infty}^{x} F_1(y)dy \leq \int_{-\infty}^{x} F_2(y)dy, \quad \forall\, x \in R = (-\infty, \infty).$$

For the proof of this lemma refer to [13].

3. Estimators of the Integrated Distribution Functions

Let $X_{i1}, X_{i2}, \ldots, X_{in_i}, i = 1, 2, \ldots, k$, be independent random samples from k populations $F_1, F_2, \ldots, F_k$, respectively. Denote the left and right endpoints of the support of F_i by c_i and $d_i, i = 1, 2, \ldots, k$.

We assume that the c.d.f.'s $F_i, i = 1, 2, \ldots, k$, are continuous, and satisfy monotone inequality

$$F_1 \geq_{\mathrm{SSD}} F_2 \geq_{\mathrm{SSD}} \cdots \geq_{\mathrm{SSD}} F_k. \tag{1}$$

In addition, assume that the populations satisfy the condition of Lemma 1 and have finite second moments; furthermore $-\infty < c_i < d_i < \infty$.

From Lemma 1, inequality (1) is equivalent to

$$\int_{-\infty}^{x} F_1(y)dy \leq \int_{-\infty}^{x} F_2(y)dy \leq \cdots \leq \int_{-\infty}^{x} F_k(y)dy, \quad \forall\, x \in R.$$

Denote the integrated distribution functions by

$$G_i(x) = \int_{-\infty}^{x} F_i(y)dy, \quad \forall\, x \in R, \quad i = 1, 2, \ldots, k.$$

Then inequality (1) can be represented as

$$G_1(x) \leq G_2(x) \leq \cdots \leq G_k(x), \quad \forall\, x \in R. \tag{2}$$

Let $\hat{F}_i(x)$ be the empirical distribution function obtained from the ith-sample. Substituting $F_i(x)$ with $\hat{F}_i(x)$ in the definition of $G_i(x)$, we get an estimator of $G_i(x)$, denoted by $\hat{G}_i(x)$, that is $\hat{G}_i(x) = \int_{-\infty}^{x} \hat{F}_i(y)dy, \forall\, x \in R, i = 1, 2, \ldots, k$.

Define the following vectors:

$$\hat{F}(x) = (\hat{F}_1(x), \hat{F}_2(x), \ldots, \hat{F}_k(x))', \quad \forall\, x \in R,$$

$$\hat{G}(x) = (\hat{G}_1(x), \hat{G}_2(x), \ldots, \hat{G}_k(x))', \quad \forall\, x \in R.$$

Furthermore, for any $x \in [c_i, d_i]$, let:

$$U_{ij}(x) = (x - X_{ij}) I_{(X_{ij}, \infty)}(x) = \begin{cases} x - X_{ij}, & X_{ij} < x \\ 0, & X_{ij} \geq x \end{cases} \quad \text{for } 1 \leq j \leq n_i, \ 1 \leq i \leq k \tag{3}$$

and

$$\bar{U}_i(x) = \frac{1}{n_i} \sum_{j=1}^{n_i} U_{ij}(x). \tag{4}$$

Then for fixed i and $x, U_{ij}(x), \ j = 1, 2, \ldots, n_i$, are independent and identically distributed random variables, and it is easy to show from (3) that:

$$\bar{U}_i(x) = \int_{-\infty}^{x} (x - y) d\hat{F}_i(y) = \int_{-\infty}^{x} \hat{F}_i(y)dy = \hat{G}_i(x),$$

$$EU_{ij}(x) = \int_{-\infty}^{x} (x - y) dF_i(y) = \int_{-\infty}^{x} F_i(y)dy = G_i(x).$$

Then we have

$$\|\hat{G}_i(x) - G_i(x)\| = \left\| \int_{-\infty}^{x} \hat{F}_i(y)dy - \int_{-\infty}^{x} F_i(y)dy \right\| = \|\bar{U}_i(x) - E\bar{U}_i(x)\|, \quad (5)$$

for each i, where $\|\cdot\|$ denotes the sup norm. We need the following lemma.

Lemma 2. *Let Z be a stochastic process indexed by an interval $[a, b] \subset \bar{R}$, whose sample paths $t \to Z(t)$ are nondecreasing. If $EZ^2(a) < \infty$ and $EZ^2(b) < \infty$, then Z satisfies the law of large number and the central limit theorem in $\ell^\infty[a, b]$. More precisely, if $Z_1, Z_2, \ldots$ are independent identical distribution copies of Z, then it holds that*:

(1) $\|\frac{1}{n}\sum_{i=1}^{n} Z_i(t) - EZ_1(t)\| \to 0$ *a.s.*,
(2) *the sequence $n^{-\frac{1}{2}}\sum_{i=1}^{n}(Z_i - EZ_i)$ converges weakly in $\ell^\infty[a, b]$ to a tight Gaussian process.*

The first conclusion is easily obtained by [16, Theorem 19.4], and for the proof of the central limit theorem part refer to [17, Example 2.11.16].

From Lemma 2 and Eqs. (4) and (5) we immediately have the following.

Theorem 1.

$$P[\|\hat{G}_i(x) - G_i(x)\| \to 0, \textit{as } n_i \to \infty, i = 1, 2, \ldots, k] = 1.$$

Theorem 1 *shows that the estimators are strong uniformly consistent. We now derive the asymptotic distributions of the estimators.*

For convenience, let:

$$n = \sum_{i=1}^{k} n_i, \quad \text{and} \quad a_{in} = \frac{n_i}{n}, \quad i = 1, \ldots, k,$$

$$A_{jn} = \sum_{i=1}^{j} a_{in}, \quad 1 \le j \le k$$

and

$$c_{jn} = a_{jn}A_{j-1,n}/A_{jn}, \quad j \ge 2$$

and define:

$$\begin{aligned} Z_{in_i}(x) &= \sqrt{n_i}[\hat{G}_i(x) - G_i(x)], \\ \tilde{Z}_{in_i}(x) &= \sqrt{n}[\hat{G}_i(x) - G_i(x)] \\ &= Z_{in_i}(x)/\sqrt{a_{in}}, \quad i = 1, 2, \ldots, k, \\ Av_n[\hat{G}(x), r, s] &= \sum_{j=r}^{s} n_j\hat{G}_j(x) \Big/ \sum_{j=r}^{s} n_j. \end{aligned}$$

We have the following conclusion on the asymptotic distributions of $\tilde{Z}_{in_i}(x)$, $i = 1, 2, \ldots, k$.

Theorem 2. *Assume that*

$$\lim_{n\to\infty} a_{in} = a_i > 0, \quad i = 1, 2, \ldots, k,$$

then it holds that

$$(\tilde{Z}_{1n_1}(x), \tilde{Z}_{2n_2}(x), \ldots, \tilde{Z}_{kn_k}(x))' \xrightarrow{W} (\tilde{Z}_1(x), \tilde{Z}_2(x), \ldots, \tilde{Z}_k(x))'. \tag{6}$$

The right-hand side is a k-variate Gaussian process with independent components $\tilde{Z}_i(x)$, $i = 1, 2, \ldots, k$, and $\tilde{Z}_i(x)$ is a mean zero Gaussian process with covariance function

$$\mathrm{cov}(\tilde{Z}_i(x), \tilde{Z}_i(y)) = \mathrm{cov}(U_{i1}(x), U_{i1}(y))/a_i.$$

Proof. Because

$$Z_{in_i}(x) = \sqrt{n_i}[\hat{G}_i(x) - G_i(x)] = \sqrt{n_i}[\bar{U}_i(x) - E\bar{U}_i(x)],$$

by Lemma 2, we have $Z_{in_i}(x) \xrightarrow{W} Z_i(x), i = 1, 2, \ldots, k$, where $Z_i(x)$ is a Gaussian process with mean zero, and its covariance function is given by:

$$\begin{aligned}
\mathrm{cov}(Z_i(x), Z_i(y)) &= \mathrm{cov}(Z_{in_i}(x), Z_{in_i}(y)) \\
&= E[Z_{in_i}(x) Z_{in_i}(y)] - E Z_{in_i}(x) E Z_{in_i}(y) \\
&= n_i E[\bar{U}_i(x) - E\bar{U}_i(x)][\bar{U}_i(y) - E\bar{U}_i(y)] \\
&= \frac{1}{n_i} \sum_{j=1}^{n_i} [EU_{ij}(x)U_{ij}(y) - EU_{ij}(x)EU_{ij}(y)] \\
&= \mathrm{cov}(U_{i1}(x), U_{i1}(y)).
\end{aligned}$$

Since $Z_{in_i}(x), i = 1, 2, \ldots, k$ are independent, we have

$$(Z_{1n_1}(x), Z_{2n_2}(x), \ldots, Z_{kn_k}(x))' \xrightarrow{W} (Z_1(x), Z_2(x), \ldots, Z_k(x))'.$$

By Slutsky lemma, it holds

$$(\tilde{Z}_{1n_1}, \tilde{Z}_{2n_2}, \ldots, \tilde{Z}_{kn_k})' \xrightarrow{W} (\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_k)',$$

where $\tilde{Z}_i(x) = Z_i(x)/\sqrt{a_i}, i = 1, 2, \ldots, k$. □

4. Hypotheses Testing

Define hypothesis

$$H_0 : G_1(x) = G_2(x) = \cdots = G_k(x), \quad \forall\, x \in R$$

and

$$H_a : G_1(x) \le G_2(x) \le \cdots \le G_k(x), \quad \forall\, x \in R.$$

In this section we consider the test of H_0 against $H_a - H_0$. This problem has a rich history when $k = 2$; for example, Kolmogorov-type test [14] and a test base on L-statistics [11] were proposed. However, for the case of $k > 2$, although one could make a test by pairwise comparisons with the methods for $k = 2$, it may be conservative. In this section, we extend the idea of Hogg [6] to construct a test of H_0 against $H_a - H_0$ for the cases of $k \geq 2$ based on the estimators given in Sec. 3.

Define test statistic T_n by

$$T_n = \max_{2 \leq i \leq k} \sup_x T_{in}(x),$$

where

$$T_{in}(x) = \frac{\sqrt{n}\sqrt{c_{in}}[\hat{G}_i(x) - Av_n[\hat{G}(x); 1, i-1]]}{S_i(x)}$$

and

$$S_i^2(x) = \frac{1}{n_i}\sum_{k=1}^{n_i}(U_{ik}(x) - \bar{U}_i(x))^2.$$

It is easy to see that $\sup_x T_{in}(x)$ is a test statistic for $H_{0i} : G_1 = \cdots = G_i$ against $H_{ai} - H_{0i}$, where H_{ai} is given by $G_1 = \cdots = G_{i-1} \leq G_i$, $i = 2, \ldots, k$. Thus T_n may be viewed as a combination of the series of the test statistics. This is similar to the test of Hogg [6] for the homogeneity of multiple distributions. In addition, we notify that $Av_n[\hat{G}(x); 1, i-1]$ is the empirical distribution function of the sample pooled with the first $i-1$ samples, and one may expect it is a more powerful test for H_{0i} than that using the $(i-1)$th-sample only.

Let

$$a = (a_1, a_2, \ldots, a_k)', \quad \tilde{Z}(x) = (\tilde{Z}_1(x), \tilde{Z}_2(x), \ldots, \tilde{Z}_k(x))',$$

$$c_j = \lim_{n\to\infty} c_{jn} = a_j A_{j-1}/A_j,$$

where $\tilde{Z}_j(x)$ and a_j are defined as in Sec. 3, and $A_j = \sum_{i=1}^{j} a_i$. Theorem 3 gives the asymptotic null distribution of T_n.

Theorem 3. *Under the conditions of Theorem* 2, *if H_0 is true*, *then it holds that*

$$T_n \xrightarrow{W} T, \tag{7}$$

where

$$T = \max_{2 \leq i \leq k} \sup_x T_i(x)$$

and

$$T_i(x) = \frac{\sqrt{c_i}[\tilde{Z}_i(x) - Av_n[\tilde{Z}(x); 1, i-1]]}{\sigma_i(x)}, \quad i = 2, \ldots, k,$$

$\sigma_i(x) = \mathrm{Var}(U_{il}(x))$.

Proof. Under $H_0, T_{in}(x)$ can be rewritten as

$$T_{in}(x) = \frac{\sqrt{c_{in}}[\tilde{Z}_{in_i}(x) - Av_n[\tilde{Z}_n(x); 1, i-1]]}{S_i(x)},$$

where $\tilde{Z}_n(x) = (\tilde{Z}_{1n_1}(x), \tilde{Z}_{2n_2}(x), \ldots, \tilde{Z}_{kn_k}(x))'$. By Theorem 2 it is obtained that $(T_{2n}(x), T_{3n}(x), \ldots, T_{kn}(x))$ converges weakly to $(T_2(x), T_3(x), \ldots, T_k(x))$. According to the continuous mapping theorem we get Eq. (7) immediately. □

We shall reject H_0 when T_n is large. To get the critical value of the test, we first show that $T_j(x), j = 1, 2, \ldots, k$ are independent. In fact, for any $x, y \in R$ and $i > j$, it holds under H_0 that:

$$\begin{aligned}
\operatorname{cov}(T_i(x), T_j(y)) &= E[T_i(x)T_j(y)] \\
&= E\left\{ \frac{\sqrt{c_i}\left[\tilde{Z}_i(x) - \sum_{s=1}^{i-1} a_s \tilde{Z}_s(x) \Big/ \sum_{s=1}^{i-1} a_s\right]}{\sigma_i(x)} \right. \\
&\qquad \left. \cdot \frac{\sqrt{c_j}\left[\tilde{Z}_j(y) - \sum_{t=1}^{j-1} a_t \tilde{Z}_t(y) \Big/ \sum_{t=1}^{j-1} a_t\right]}{\sigma_j(y)} \right\} \\
&= \frac{\sqrt{c_i c_j}}{\sigma_i(x)\sigma_j(y)} \left\{ \left[-a_j E\tilde{Z}_j(x)\tilde{Z}_j(y) \Big/ \sum_{s=1}^{i-1} a_s \right] \right. \\
&\qquad \left. + \sum_{t=1}^{j-1} a_t^2 E\tilde{Z}_t(x)\tilde{Z}_t(y) \Big/ \left[\sum_{t=1}^{j-1} a_t \sum_{s=1}^{i-1} a_s \right] \right\} \\
&= \frac{\sqrt{c_i c_j}}{\sigma_i(x)\sigma_j(y)} \left\{ \left[-EZ_j(x)Z_j(y) \Big/ \sum_{s=1}^{i-1} a_s \right] \right. \\
&\qquad \left. + \sum_{t=1}^{j-1} a_t EZ_t(x)Z_t(y) \Big/ \left[\sum_{t=1}^{j-1} a_t \sum_{s=1}^{i-1} a_s \right] \right\} \\
&= \frac{\sqrt{c_i c_j}}{\sigma_i(x)\sigma_j(y)\sum_{s=1}^{i-1} a_s} \left\{ -EZ_j(x)Z_j(y) \right. \\
&\qquad \left. + \sum_{t=1}^{j-1} a_t EZ_t(x)Z_t(y) \Big/ \sum_{t=1}^{j-1} a_t \right\} = 0.
\end{aligned}$$

Noting that $T_j(x), j = 1, \ldots, k$ are Gaussian processes, we know they are independent to each other.

Furthermore, we may also compute under H_0 that:

$$\begin{aligned}
\operatorname{cov}(T_i(x), T_i(y)) &= E[T_i(x)T_i(y)] \\
&= E\left\{ \frac{\sqrt{c_i}}{\sigma_i(x)} \left[\tilde{Z}_i(x) - \sum_{s=1}^{i-1} a_i \tilde{Z}_s(x) \Big/ \sum_{s=1}^{i-1} a_s \right] \right. \\
&\qquad \left. \times \frac{\sqrt{c_i}}{\sigma_i(y)} \left[\tilde{Z}_i(y) - \sum_{t=1}^{i-1} a_t \tilde{Z}_t(y) \Big/ \sum_{t=1}^{i-1} a_t \right] \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{c_i}{\sigma_i(x)\sigma_i(y)} \\
&\quad \times \left\{ E[\tilde{Z}_i(x)\tilde{Z}_i(y)] + \sum_{t=1}^{i-1} a_t^2 E[\tilde{Z}_t(x)\tilde{Z}_t(y)] \Bigg/ \left(\sum_{t=1}^{i-1} a_t\right)^2 \right\} \\
&= \frac{c_i}{\sigma_i(x)\sigma_i(y)} \\
&\quad \times \left\{ E[Z_i(x)Z_i(y)]/a_i + \sum_{t=1}^{i-1} a_t E[Z_t(x)Z_t(y)] \Bigg/ \left(\sum_{t=1}^{i-1} a_t\right)^2 \right\} \\
&= \frac{c_i}{\sigma_i(x)\sigma_i(y)} E[Z_1(x)Z_1(y)] \\
&= \frac{\operatorname{cov}(U_{i1}(x), U_{i1}(y))}{\sigma_i(x)\sigma_i(y)}.
\end{aligned}
$$

Thus $T_j(x)$, $j = 2, \ldots, k$ are independent Gaussian processes with mean zeros and covariance functions given by $\operatorname{cov}(T_i(x), T_i(y)) = \operatorname{cov}(U_{i1}(x), U_{i1}(y))/\sigma_i(x)\sigma_i(y)$. Note $\operatorname{Var}(T_i(x)) = 1$.

From the above properties of $T_j(x)$s and Borell inequality (see, for example, [15, Lemma 4], it holds that:

$$
\begin{aligned}
P(T \geq t) &= 1 - P\left(\sup_x T_j(x) < t, 2 \leq j \leq k\right) \\
&= 1 - \prod_{j=2}^{k} P\left(\sup_x T_j(x) < t\right) \leq 1 - [1 - 2(1 - \Phi(t))]^{k-1} \\
&= 1 - [2\Phi(t) - 1]^{k-1}. \qquad (8)
\end{aligned}
$$

Consequently, the critical value of the test can be obtained by letting the right-hand side of inequality (8) be equal to a given significant level.

Remark. Without the assumption that the supports of the distributions are finite intervals, the conclusion of Theorem 2 may not hold. However, the conclusion of Theorem 3 still keeps true. In fact, T_{in} only depends on the differences $\hat{G}_i(x) - \hat{G}_j(x)$, which must satisfy the conditions of the central limit theorem of empirical processes when H_0 is true.

5. Simulation Study

In order to evaluate the performance of the proposed test, we carry out simulations with $k = 3$.

In the simulation experiments, we consider the following five scenarios, where F_1, F_2 and F_3 denote the three populations or distributions considered.

(I) $F_1 = F_2 = F_3 = U[1, 2]$, where $U[a, b]$ is the uniform distribution on $[a, b]$.

(II) $F_1 = U[0, 1.1], U[0, 1.2]$ or $U[0, 1.3], F_2 = F_3 = U[0, 1]$.
(III) F_1 is taken as the exponential distribution with $\lambda = 0.9, \lambda = 0.8$ or $\lambda = 0.7$ respectively, and $F_2 = F_3$ are taken as the exponential distribution with $\lambda = 1$.
(IV) F_1 is beta distribution with $a = b = 2$, and F_2, F_3 are taken as $U[0, 1]$.
(V) F_1, F_2 are beta distributions with $a = b = 2$, and F_3 is $U[0, 1]$.

The sample sizes of the three distributions are taken as the same in each scenario, and they are set at 100, 150 and 300 in different simulations for evaluating the effect of sample size. In addition, different significant levels are considered. In Scenario (I), three significance levels, $\alpha = 0.01, 0.025$ and 0.05 are used. In other scenarios, $\alpha = 0.05$ is used. The critical values are calculated with inequality (8). For each simulation experiment, we compute the rejection frequency with 1000 replications. The simulation results are given in Tables 1–3.

From Table 1, we see that the simulated size of the proposed test is reasonable and gets closer to α with the sample size n increasing. Furthermore, from Tables 2 and 3, we could have the following observations.

(1) With the increasing of sample sizes, the power of the proposed test increases fast.
(2) The power is related to how the probability distributions go against the null hypothesis.

Table 1. Simulated sizes of the proposed test: Scenario (I).

n	α		
	0.01	0.025	0.05
100	0.017	0.022	0.046
150	0.016	0.025	0.048
300	0.013	0.026	0.051

Table 2. Simulated powers of the proposed test: Scenario (II) and (III).

n	F_1: uniform distribution on		
	$[0, 1.1]$	$[0, 1.2]$	$[0, 1.3]$
100	0.252	0.299	0.540
150	0.653	0.797	0.984
300	0.892	0.973	0.996
n	F_1: exponential distribution with		
	$\lambda = 0.9$	$\lambda = 0.8$	$\lambda = 0.7$
100	0.116	0.242	0.481
150	0.179	0.477	0.775
300	0.228	0.651	0.928

Table 3. Simulated powers of the proposed test: Scenario (IV) and (V).

Scenarios		n		
		100	150	300
F_1	beta(2, 2)			
F_2	$U[0,1]$	0.432	0.673	0.978
F_3	$U[0,1]$			
F_1	beta(2, 2)			
F_2	beta(2, 2)	0.562	0.795	0.989
F_3	$U[0,1]$			

6. Real Examples

In order to illustrate the theory developed in previous sections, we give two examples in this section.

6.1. *Example* 1

First we consider the financial data on annual total return of stocks and bonds by decade from 1810 to 1989. The data comes from Global Financial Data (www.globalfindata.com). Table 4 summarizes annual total return of stocks and bonds by decade. Decade i indicates the decade starting during the year $1810+i*10$.

Stocks and bonds are defined as the two populations respectively, for which we want to test the distribution identity against the dominance of bond population in the sense of second stochastic order. Then the test statistic $T_n = 3.098$, and the critical values corresponding to three significance levels $\alpha = 0.01, 0.025$ and 0.05 are $c(\alpha) = 2.58, 2.24, 1.96$, respectively. Obviously, $T_n > c(\alpha)$, this result indicates that the null hypothesis should be rejected. This is consistent with the result of Rojo and EI Barmi [12].

6.2. *Example* 2

Now we consider the data given in Data Set II from [7]. The data consists of the survival times for patients with carcinoma of the oropharynx and several covariates. One of the covariates is an ordinal categorical variable with four levels, which indicates increasing levels of deterioration of lymph nodes in each patient, measured

Table 4. Stocks/bonds total return by decade.

Decade	1	2	3	4	5	6	7	8	9
Stocks	2.68	5.31	4.53	6.73	0.45	15.73	7.58	6.72	5.45
Bonds	6.41	5.92	6.25	4.89	5.35	6.65	7.96	5.53	3.92
Decade	10	11	12	13	14	15	16	17	18
Stocks	9.62	4.69	13.86	−0.17	9.57	18.53	8.17	6.75	16.64
Bonds	2.60	2.23	5.84	4.20	2.49	0.73	1.96	4.32	13.26

Table 5. Observed survival times grouped by the severity of the lymph node deterioration.

	Survival times
Population 0	243 560 376 167 404 374 446 574 561 714 432 351 130 296 545 525 191 372 238 446 541 107 775 336 343 324 498 38 173
Population 1	275 254 81 154 532 553 347 382 599 338 763 929 301 328 1092 324 216 631 575
Population 2	184 1064 222 661 546 279 915 228 407 230 666 477 310 170 465 105 346 518 395 608 291 128
Population 3	159 219 413 274 208 174 213 407 334 112 209 369 513 757 266 317 219 637 112 99 99 461 363 293 147 726 264 11 89 11 147 696 112 308 15 172 544 800 787 205 127 370 805 192 273 548 517 74 459 494 173 1565 256 134 162 262 307 782 414 480 245 911 279 144 94 177 270 327 916 637 235 255

at time of entry in the study. Because lymph node deterioration is an indication of the seriousness of the carcinoma, it is reasonable to expect that the four survival time distributions would be stochastically ordered by the severity of the lymph node deterioration. Here we delete all censored data and reproduce the data in Table 5, where population 0 indicates no evidence of lymph node metastases, and populations 1–3 indicate the presence of sequentially more and more serious tumors.

Then the test statistic $T_n = 26.27$, and the critical value corresponding to the significance level $\alpha = 0.01$ is $c(\alpha) = 2.94$, which is much smaller than T_n. This result supports the alternative hypothesis, and is consistent with the result of Wang [18], where the comparison of categorized populations was employed.

7. Conclusions

In this paper, we focused on the second-order stochastic dominance among multiple probability distributions and proposed a test for the stochastic equality of the multiple distributions based on the empirical estimators of the integrated distribution functions. The asymptotic distribution of the proposed test statistic under the null hypothesis is derived and a method to decide the critical value is conducted. Simulation results show that the test works well in general.

Theoretically, the critical value given by inequality (8) makes the test slightly conservative. Another method for getting the critical value is bootstrap, which costs more computing time, and is not considered in this paper.

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