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Smallest confidence intervals for one binomial proportion

Weizhen Wang

Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435, USA

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Abstract

We specify three classes of one-sided and two-sided $1-\alpha$ confidence intervals with certain monotonicity and symmetry on the confidence limits for the probability of success, the parameter in a binomial distribution. For each class of one-sided confidence intervals the smallest interval, in the sense of the set inclusion, is obtained based on the direct analysis of coverage probability functions. A simple sufficient and necessary condition for the existence of the smallest two-sided confidence interval is provided and the smallest interval is derived if it exists. Thus the proposed intervals are uniformly most accurate, and have the uniformly minimum expected length as well.

Keywords: Binomial distribution; Coverage probability; False coverage probability; Set inclusion

1. Introduction

Interval estimation of one binomial proportion is one of the most basic problems in statistics theory. Let X be a binomial random variable with n trials and a binomial proportion p. The confidence interval for the parameter p is of interest in this paper. Many solutions have been proposed, however, no one is even close to being accepted by the majority of statisticians. The existing confidence intervals are basically derived by using either normal approximations or exact probability calculations on binomial distributions. For the convenience of discussion, we call these approximate intervals and exact intervals, respectively. The most widely used approximate interval for p is the Wald interval:

$$\hat{p} \pm z_{\alpha/2}(\hat{p}(1-\hat{p})/n)^{1/2},$$

E-mail address: wwang@math.wright.edu.

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where $\hat{p} = X/n$ and $z_{\alpha/2}$ is the upper $100\alpha/2$th percentile of the standard normal distribution. Correspondingly, the one-sided confidence intervals for p are

$$[0, \hat{p} + z_\alpha(\hat{p}(1-\hat{p})/n)^{1/2}],$$

and

$$[\hat{p} - z_\alpha(\hat{p}(1-\hat{p})/n)^{1/2}, 1].$$

According to a thorough study by Brown et al. (2001, 2002), as well as others, for instance, Santner (1998), Agresti and Coull (1998) and Newcombe (1998), this interval should not be used anymore because its coverage probability can be erratically poor even when p is not close to zero or one. Alternatively, Brown et al. (2001, 2002) recommended three approximate intervals, the Wilson (1927) interval, the Agresti–Coull (1998) interval and Jeffreys' equal-tailed interval. The Agresti–Coull (1998) interval is the simplest one among the three and has a similar form to the Wald interval. However, although the coverage probabilities of the three intervals are much closer to the nominal level $1-\alpha$ than the Wald interval, like all approximate intervals, their confidence coefficients defined as the minimum coverage probability are less than $1-\alpha$. There have been many efforts to obtain exact intervals. Since the calculation is based on the binomial probability, the resultant intervals typically have confidence levels at least $1-\alpha$. Here we mention five. Clopper and Pearson (1934) proposed the first exact two-sided interval using the equal-tail rule. The interval construction is very simple, but this commonly used $1-\alpha$ Clopper–Pearson interval is very conservative, i.e., the confidence coefficient may be much higher than $1-\alpha$. We will show later, however, that the Clopper–Pearson interval can be the "best" interval if being used appropriately. The Crow (1956) interval is the first one with a proven optimality: this interval minimizes the total length of intervals over all observed values of X. The interval is obtained by inverting the acceptance regions for a family of tests. Later, Blyth and Still (1983) proposed an interval which minimizes the total length, like the Crow interval. Their interval is also obtained by inverting tests, but different criteria are used to choose the acceptance regions. A "left-hand" Crow interval was mentioned in Blyth and Still (1983, p. 110). In some cases of n and α the Crow (1956) interval and the Blyth and Still (1983) interval coincide (see when they are identical in the Discussion section). Casella (1986) proposed a refining technique to uniformly improve any existing interval.

The standard procedure to find the "best" confidence interval is to specify a class of confidence intervals and then seek the "best" one in this class. The criteria of minimizing the expected length or the false coverage probability for the interval may be used to obtain the "best" interval. The class should include enough intervals. However, if the class is too big, then typically there does not exist the "best" interval. Randomized confidence intervals are not preferred since they are not used in practice. The confidence level is the most important feature of a confidence interval. It measures the chance that the interval captures the parameter of interest. In this paper, we focus on the construction of $1-\alpha$ confidence intervals $[L(X), U(X)]$ for the parameter p, i.e.,

$$\inf_p P_p(p \in [L(X), U(X)]) \geqslant 1-\alpha, \tag{1.1}$$

and the interval is said to have a confidence level $1-\alpha$. If one of the confidence limits is a constant, it is one-sided. For example, if one wants to establish that the defective rate of a new product is less than a certain amount, then the interval of the form $[0, U(X)]$ is of interest. If both limits are random variables, then it is two-sided. The function $P_p(p \in [L(X), U(X)])$ is the coverage probability of the interval $[L(X), U(X)]$. Due to the discreteness, the construction of the "best" nonrandomized confidence interval for p is challenging. Such interval does not exist if one specifies an inappropriate class of intervals. The first goal of the paper is to derive the smallest one-sided $1-\alpha$ confidence interval for p. Here, the smallest interval means that on each value of X it is *a subset of any other interval* in the specified class of intervals. Therefore, this interval is the "best" in the strongest sense, and it automatically minimizes the expected length or the false coverage probability uniformly. A much more challenging problem is to derive the smallest two-sided interval, which in fact does not always exist. For this, we discuss when such interval exists and how to calculate the smallest interval provided its existence. In Section 3 the smallest one-sided $1-\alpha$ confidence intervals are presented. Section 4 contains a sufficient and necessary condition for the existence of the smallest two-sided $1-\alpha$ interval. All proofs are postponed to the appendix.

2. A general theorem

Suppose X is a discrete random variable with a probability mass function $p(x;\theta)$ and θ is a one-dimensional parameter. Let S and ϑ be the sample space and the parameter space of X, respectively. We discuss which parameter values are included in a confidence set or interval for θ.

Theorem 1. *Let $C(X)$ be a confidence set of the parameter θ with an observation X. For each $x \in S$,*

$$C(x) = \{\theta \in \vartheta : p(x;\theta) \text{ is used to compute the coverage probability of } C(X)\}. \tag{2.1}$$

See the proof in the appendix.

Corollary 1. *For a confidence interval $I = [L(X), U(X)]$, and for each $x \in S$, the term $p(x;\theta)$ is used to compute the coverage probability of $[L(X), U(X)]$, denoted by $f_I(\theta)$, if and only if $L(x) \leqslant \theta \leqslant U(x)$. Thus*

$$L(x) = \text{the smallest } \theta \text{ where } p(x;\theta) \text{ is used to compute the coverage probability}. \tag{2.2}$$

$$U(x) = \text{the largest } \theta \text{ where } p(x;\theta) \text{ is used to compute the coverage probability}. \tag{2.3}$$

This simple corollary provides an important guidance on how to select the smallest confidence interval. In the binomial case, $\theta = p$, suppose the one-sided $1-\alpha$ interval of the form $[0, U(X)]$ is of interest. For each fixed $x \in S$, one should exclude the term $p(x,\theta)$ from $f_I(\theta)$ when θ is as small as possible while keeping $f_I(\theta)$ no less than $1-\alpha$. Eq. (3.1), for example, is obtained following this fact.

3. Smallest one-sided confidence intervals

From now on, suppose X is a binomial random variable with n trials and an unknown success probability p and $\text{bin}(x;n,p)$ and $F_B(x;n,p)$ are the probability mass function (pmf) and the cumulative distribution function (cdf), respectively, for X. The sample space S_B contains integers

$0, \ldots, n$, and the parameter space $\vartheta_B = [0, 1]$. We now focus on a class of $1 - \alpha$ confidence intervals for p, $\mathscr{B}_u$, satisfying the following:

(a) $L(x) = 0$; and
(b) $0 \leqslant U(x) \leqslant U(x+1) \leqslant 1$ for any $x \in S_B$.

i.e., the one-sided nondecreasing confidence intervals to estimate the upper limit of the proportion p. The monotonicity in the upper limits is reasonable because the success probability p tends to be bigger if more successes are observed. The "best" interval is defined as follows.

Definition 1. For a class of $1-\alpha$ confidence intervals $\mathscr{A}$, an interval $[A(X), B(X)]$ is called the smallest confidence interval (in $\mathscr{A}$) if

(i) $[A(X), B(X)]$ belongs to $\mathscr{A}$,
(ii) $[A(X), B(X)]$ is the intersection of all intervals in $\mathscr{A}$.

The smallest confidence interval is equivalent to the most accurate confidence interval proposed by Bol'shev (1965) (which is different from the one by Lehmann, 1986, p. 90), as defined below.

Definition 2. For a class of $1-\alpha$ confidence intervals $\mathscr{A}$, an interval $[A(X), B(X)]$ is called the most accurate confidence interval (in $\mathscr{A}$) if

(i) $[A(X), B(X)]$ belongs to $\mathscr{A}$,
(ii) If the confidence limits of interval $[A^*(X), B^*(X)]$ is nondecreasing in X, and if the set difference $[A(X), B(X)] - [A^*(X), B^*(X)]$ is not empty for some value of X, then the confidence coefficient of interval $[A^*(X), B^*(X)]$ is strictly less than $1-\alpha$.

As pointed out by a referee, the smallest and most accurate confidence intervals are identical, however, the former seems to be a more direct definition. Next we derive the smallest interval with the smallest upper limit, $U_B(X)$, in $\mathscr{B}_u$.

Theorem 2. *For* $0 < \alpha < 1$ *and each* $x(< n) \in S_B$, *let* $U_B(x)$ *be the solution, with respect to* p, *of*

$$F_B(x; n, p) = \alpha; \tag{3.1}$$

for $x = n$, *let* $U_B(x) = 1$. *Then the interval* $[0, U_B(X)]$ *is the smallest one in* $\mathscr{B}_u$.

The theorem was first given by Bol'shev (1965) and a different proof can be found in Wang (2002). In fact Bol'shev (1965) constructed the most accurate confidence interval whenever X has a cdf $F(x, p)$, which is monotone with respect to the parameter p. Therefore the interval given in Theorem 2 is the smallest because it is the most accurate. For more details of Bol'shev's method, see Bagdonavicius et al. (1997) and Nikulin (1990).

Remark 1. If one uses the Clopper–Pearson (1934) method, which is typically used to construct two-sided intervals (very conservative), to obtain an one-sided interval, it is equal to the proposed. However, the "smallest" or "most accurate" property, is seldom mentioned in practice.

One can easily construct the smallest confidence interval for the other one-sided $1-\alpha$ confidence interval class:

$$\mathscr{B}_l = \{[L(X), 1] : 0 \leqslant L(x) \leqslant L(x+1)\}. \tag{3.2}$$

Let

$$L_B(X) = 1 - U_B(n - X). \tag{3.3}$$

Then the interval $[L_B(X), 1]$ is the smallest confidence interval in $\mathscr{B}_l$. Also for $x > 0$,

$$F_B(x-1; n, L_B(x)) = 1 - \alpha. \tag{3.4}$$

4. Smallest two-sided confidence intervals

The main reason that the smallest one-sided intervals always exist is that their upper limits never cross with the lower limits (note $0 \leqslant U(X)$ or $L(X) \leqslant 1$). For the two-sided interval, this seldom happens. If the crossover is not severe, the smallest interval may still exist as shown below. We search for the smallest confidence interval in the following class, denoted by $\mathscr{B}$, of two-sided $1-\alpha$ confidence intervals of the form $[L(X), U(X)]$ satisfying:

(a) $0 \leqslant L(x) \leqslant L(x+1) \leqslant 1$ and $0 \leqslant U(x) \leqslant U(x+1) \leqslant 1$;
(b) $L(x) = 1 - U(n-x)$,

for any $x \in S_B$. The second condition is natural because $n-X$ also follows a binomial distribution with a binomial proportion $1-p$.

Theorem 3. *For* $0 < \alpha < 1$, *suppose*

$$F_B(0; n, 0.5) \geqslant \alpha \tag{4.1}$$

then the interval $[L_B(X), U_B(X)]$, *where* $L_B(X)$ *and* $U_B(X)$ *are given in* (3.3) *and* (3.1), *respectively*, *is the smallest one in* $\mathscr{B}$.

See the appendix for the proof.

Remark 2. The interval $[L_B(X), U_B(X)]$ is indeed the $1-2\alpha$ Clopper–Pearson (1934) interval. Theorem 3 provides a sufficient condition, (4.1), for the $1-2\alpha$ Clopper–Pearson interval to be the smallest $1-\alpha$ confidence interval in $\mathscr{B}$. In fact, (4.1) is also a necessary condition which follows the next result.

Theorem 4. *If* (4.1) *is not true*, *then the interval* $[L_B(X), U_B(X)]$ *does not belong to* $\mathscr{B}$.

See the appendix for the proof.

To summarize the results from the previous two theorem, we have

Corollary 2. *The* $1-2\alpha$ *Clopper–Pearson interval* $[L_B(X), U_B(X)]$ *is the smallest interval in* $\mathscr{B}$ *if and only if* (4.1) *is true*.

Remark 3. Condition (4.1) is equivalent to $U_B(0) \geqslant L_B(n)$ or $U_B(0) \geqslant 0.5$. So the $1-2\alpha$ Clopper–Pearson interval is the smallest interval in $\mathscr{B}$ if and only if its lower confidence limits never cross with its upper limits.

A natural question is then raised: what happens if $U_B(0)$ is less than some of the lower limits? In other words, are there cases for n and α where the smallest interval in $\mathscr{B}$ exists but is different from $[L_B(X), U_B(X)]$? The answer is "Yes".

Theorem 5. *For* $0<\alpha<1$, *let* a_0 *be the smallest solution of* $g(p)=1-\alpha$, *where*

$$g(p) = 1 - \text{bin}(0; n, p) - \text{bin}(n; n, p), \tag{4.2}$$

if solutions exist, and let a_0 *be* 0.5, *otherwise. Let*

$$U_{B1}(x) = \begin{cases} a_0 & \text{if } x = 0, \\ U_B(x) & \text{if } x > 0, \end{cases} \tag{4.3}$$

and let

$$L_{B1}(X) = 1 - U_{B1}(n - X). \tag{4.4}$$

Suppose

$$F_B(0; n, 0.5) < \alpha \ (\textit{equivalently } U_B(0) < L_B(n)), \tag{4.5}$$

and

$$U_{B1}(0) + U_{B1}(1) \geqslant 1 \ (\textit{equivalently } U_{B1}(0) \geqslant L_{B1}(n-1)) \tag{4.6}$$

then the interval $[L_{B1}(X), U_{B1}(X)]$ *is the smallest one in* $\mathscr{B}$.

See the appendix for the proof.

Remark 4. Theorem 5 implies that the smallest confidence interval exists in $\mathscr{B}$ if an upper limit, $U_B(0)$, of the $1-2\alpha$ Clopper–Pearson interval cross with its lower limits only one time, i.e. $L_B(n-1) \leqslant U_B(0) < L_B(n)$ because it implies (4.6). However, the smallest interval is not the $1-2\alpha$ Clopper–Pearson interval. If there are two crossovers, i.e., $L_B(n-2) \leqslant U_B(0) < L_B(n-1)$, the smallest interval may still exist, but not always, and the smallest interval again is not the $1-2\alpha$ Clopper–Pearson interval. We give an example below.

Example 1. Suppose $n=5$ and $\alpha=0.085$. Now $U_B(0)+U_B(1)=0.3893+0.6027<1$, implying $U_B(0)<L_B(4)$. But $U_{B1}(0)=a_0=0.4060$ is no less than $L_{B1}(4)$ and the smallest interval in $\mathscr{B}$ exists and is obtained following (4.3) and (4.4):

x	0	1	2	3	4	5
$U_{B1}(x)$	0.4060	0.6027	0.7685	0.8976	0.9824	1
$L_{B1}(x)$	0	0.0176	0.1024	0.2315	0.3973	0.5940
$U_B(x)$	0.3893	0.6027	0.7685	0.8976	0.9824	1
$L_B(x)$	0	0.0176	0.1024	0.2315	0.3973	0.6107

To summarize the results in Theorems 3 and 5, we have the following.

Corollary 3. *A sufficient condition for the existence of the smallest interval in the class $\mathscr{B}$ is* (4.6), *and the confidence limits of the smallest interval is given in either* (3.1) *and* (3.3) *or* (4.3) *and* (4.4) *depending on whether* $U_B(0) \geqslant 0.5$.

See the appendix for the proof.

Example 1 suggests that the existence of the smallest interval really depend on the number of times that $U_{B1}(0)$ crosses with the lower limits, $L_{B1}(x)$'s, but not for $U_B(0)$. Indeed, if $U_{B1}(0)$ crosses with the lower limits $L_{B1}(x)$ for more than one time, the smallest interval does not exist.

Theorem 6. *If* (4.6) *is not true*, *then the smallest interval in the class $\mathscr{B}$ does not exist*.

The proof is lengthy and is postponed to the appendix. To summarize Corollary 3 and Theorem 6, we have the following.

Corollary 4. *The smallest* $1-\alpha$ *confidence interval exists in the class $\mathscr{B}$ if and only if* (4.6) *is true*.

5. Discussion

In this paper, we use the set inclusion to decide whether one interval is superior than another. For the one-sided interval, the solution is final since the smallest interval always exists and cannot be improved any further. Therefore, the Clopper–Pearson method indeed yields the best one-sided interval. For the two-sided interval we provide a precise condition under which the smallest interval exists for one proportion. This condition is of more theoretical interest because the smallest interval exists only when n or α is quite small. The nonexistence, however, may explain why statisticians after several decades still cannot reach an agreement about which is the "best" interval. The Crow (1956) interval is equal to the Blyth and Still (1983) interval if the smallest interval exists and they are typically different otherwise. Nevertheless, it seems reasonable using the set inclusion to define a "better" interval, especially for one-sided intervals. A much more interesting problem for the future research is how to construct the smallest interval when there exist nuisance parameters.

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Appendix A

Proof of Theorem 1. The coverage probability of $C(X)$, denoted by $f_C(\theta)$, is

$$f_C(\theta) = P_\theta(\theta \in C(X)) = \sum_{x \in S} I_{\{\theta \in C(x)\}} p(x;\theta), \tag{A.1}$$

where I, as a function of θ, is the indicator function of the set $C(x)$. For each fixed x, the term $p(x;\theta)$ is included in the above summation if and only if $\theta \in C(x)$. $\square$

Proof of Theorem 3. By the definition of $U_B(X)$ and (4.1), it is clear that $U_B(x)$ is increasing in x and $U_B(0) \geqslant 0.5$. Therefore, $L_B(x)$ is increasing in x and $L_B(n) \leqslant 0.5$ by (3.3). So $U_B(0)$ does not cross with any of the lower limits, $L_B(x)$'s. Therefore, if we let $f_{B'}(p)$ be the coverage probability of $[L_B(X), U_B(X)]$, it is easy to show that on [0.5, 1]

$$f_{B'}(p) = \begin{cases} 1 & \text{if } 0.5 \leqslant x \leqslant U_B(0), \\ f_B(p) & \text{if } x > U_B(0), \end{cases} \tag{A.2}$$

where $f_B(p)$ is the coverage probability of $[0, U_B(X)]$, no smaller than $1-\alpha$ as shown in Theorem 2. Then $f_{B'}(p)$ is also no smaller than $1-\alpha$ and $[L_B(X), U_B(X)]$ is in $\mathscr{B}$.

Suppose $[L_B(X), U_B(X)]$ is not the smallest in $\mathscr{B}$. Then there exists an interval $[L_0(X), U_0(X)] \in \mathscr{B}$ and a $x_0 \in S_B$ so that either

(1)

$$L_0(x_0) > L_B(x_0),$$

or

(2)

$$U_0(x_0) < U_B(x_0).$$

Without loss of generality, suppose (1) is the case. Since the coverage probability of $[L_0(X), 1]$ is no smaller than that of $[L_0(X), U_0(X)]$, $[L_0(X), 1]$ belongs to $\mathscr{B}_l$, which contradicts with the fact that $[L_B(X), 1]$ is the smallest one in $\mathscr{B}_l$. $\square$

Proof of Theorem 4. It suffices to show that $[L_B(X), U_B(X)]$ has a coverage probability less than $1-\alpha$ for some p_0. Since (4.1) is not true, $U_B(0) < 0.5 < L_B(n)$. Since $F_B(0; n, U_B(0)) = \alpha$ and $c = \text{bin}(n; n, U_B(0))$ is positive, one can choose a small positive Δp so that

$$F_B(0; n, U_B(0) + \Delta p) > \alpha - c/3 \quad \text{and} \quad \text{bin}(n; n, U_B(0) + \Delta p) > 2c/3.$$

Let $p_0 = U_B(0) + \Delta p$. Then

$$f_B(p_0) \leqslant \sum_{x=1}^{n-1} \text{bin}(x; n, p_0) = 1 - F_B(0; n, p_0) - \text{bin}(n; n, p_0) < 1 - \alpha. \qquad \square$$

Lemma 1. *For any confidence interval* $[L(X), U(X)]$ *in the class* $\mathscr{B}$, $L(0) = 0$.

Proof. Suppose $L(0) > 0$. When $p < L(0)$, $I_{\{p \geqslant L(x)\}}$ is zero for any $x \in S_B$. Then, the coverage probability of $[L(X), U(X)]$,

$$\begin{aligned} f(p) &= P_p(L(X) \leqslant p \leqslant U(X)) \leqslant P_p(L(X) \leqslant p) \\ &= \sum_{x=0}^{n} I_{\{p \geqslant L(x)\}} \,\text{bin}(x; n, p) = 0, \end{aligned} \tag{A.3}$$

is not always at least $1-\alpha$, a contradiction. $\square$

Proof of Theorem 5. Since $g(p)$ is symmetric about 0.5, $U_{B1}(0)\leqslant 0.5$. Due to (4.6), $U_{B1}(1)\geqslant 0.5$. So $U_{B1}(x)$ is nondecreasing. Also due to (4.6), $L_{B1}(n-1)\leqslant U_{B1}(0)$. The ending points of the interval $[L_{B1}(X), U_{B1}(X)]$ in [0, 0.5] are listed as

$$L_{B1}(0)(=0=L_B(0)), \ldots, L_{B1}(n-1)(=L_B(n-1)), U_{B1}(0)$$

in a nondecreasing order. Note that $L_{B1}(n)=0.5$ if $U_{B1}(0)=0.5$. Let

$$I_i=[L_{B1}(i), L_{B1}(i+1)), i=0,\ldots,n-2,$$
$$I_{n-1}=[L_{B1}(n-1), U_{B1}(0)), I_n=[U_{B1}(0), 0.5].$$

Then $\{I_i\}_{i=0}^{n}$ form a partition of [0, 0.5]. Let $f_{B1}(p)$ be the coverage probability of $[L_{B1}, U_{B1}]$. On I_i for $i\leqslant n-2$, since $L_{B1}(i)\leqslant p<L_{B1}(i+1)\leqslant U_{B1}(y)$ for all $y\in S_B$, by Corollary 1, the terms $\mathrm{bin}(x; n, p)$ for $x<i+1$ are used to compute $f_{B1}(p)$, and

$$\begin{aligned} f_{B1}(p) &= \sum_{x=0}^{i} \mathrm{bin}(x; n, p) = F_B(i; n, p) \geqslant F_B(i; n, L_{B1}(i+1)) \\ &= F_B(i; n, L_B(i+1)) = 1-\alpha \end{aligned} \tag{A.4}$$

by (3.4) and that $F_B(x; n, p)$ is nonincreasing in p. On I_{n-1}, since $L_B(n)>0.5$ due to (4.5), I_{n-1} is a subset of $[L_B(n-1), L_B(n))$ and f_{B1} is no less than $1-\alpha$. From (4.2) and (4.3), it is clear that $g(U_{B1}(0))=1-\alpha$ if $U_{B1}(0)<0.5$. Since $g'(p)=n(1-p)^{n-1}-np^{n-1}>0$ when $p<0.5$, $g(p)$ is strictly increasing on I_n. If $U_{B1}(0)<0.5$,

$$f_{B1}(p)\geqslant \sum_{x=1}^{n-1} \mathrm{bin}(x; n, p) = g(p) \geqslant g(U_{B1}(0)) = 1-\alpha$$

on I_n; if $U_{B1}(0)=0.5$, then $L_{B1}(n)=0.5$ and

$$f_{B1}(p)=\sum_{x=0}^{n} \mathrm{bin}(x; n, p) = 1 \geqslant 1-\alpha$$

on I_n. Therefore, $[L_{B1}(X), U_{B1}(X)]$ belongs to $\mathscr{B}$ under (4.5) and (4.6).

If $[L_{B1}(X), U_{B1}(X)]$ is not the smallest in $\mathscr{B}$, then there exists an interval $[L_1(X), U_1(X)]\in\mathscr{B}$ and a $x_0\in S_B$ so that either

(1)

$$L_1(x_0)>L_{B1}(x_0) \quad \text{and} \quad L_1(x_0-1)<L_1(x_0),$$

or

(2)

$$U_1(x_0)<U_{B1}(x_0) \quad \text{and} \quad U_1(x_0)<U_1(x_0+1).$$

Without loss of generality, suppose (1) is the case. Then $x_0 > 0$ due to Lemma 1. Let f_1 be the coverage probability of the interval $[L_1(X), U_1(X)]$. There are two possibilities:

(A) If $x_0 < n$, for any $p \in (\max(L_1(x_0-1), L_{B1}(x_0)), L_1(x_0))$ (not an empty interval).

$$\begin{aligned} f_1(p) &\leqslant \sum_{x=0}^{x_0-1} \mathrm{bin}(x; n, p) = F_B(x_0-1; n, p) < F_B(x_0-1; n, L_{B1}(x_0)) \\ &= 1-\alpha \end{aligned}$$

(the above strict inequality is because $F_B(x; n, p)$ is strictly decreasing in p for $0 \leqslant x < n$, and also note $L_{B1}(x_0) = L_B(x_0)$ for $x_0 < n$) a contradiction.

(B) If $x_0 = n$, then $L_1(n) > L_{B1}(n) \geqslant U_{B1}(0) > U_1(0)$. For any $p \in (U_1(0), U_{B1}(0))$,

$$f_1(p) \leqslant \sum_{x=1}^{n-1} \mathrm{bin}(x; n, p) = g(p) < g(U_{B1}(0)) \leqslant 1-\alpha,$$

a contradiction as well.

Therefore, $[L_{B1}(X), U_{B1}(X)]$ is the smallest under (4.5) and (4.6). □

Proof of Corollary 3. It suffices to show that (4.1) implies (4.6). If (4.1) is true, then $U_B(0) \geqslant 0.5$ and $\mathrm{bin}(0; n, p)$ is no smaller than α on $[0, 0.5]$. Therefore,

$$g(p) \leqslant 1 - \mathrm{bin}(0; n, p) \leqslant 1 - F_B(0; n, U_B(0)) \leqslant 1-\alpha$$

on $[0, 0.5]$. Note that $g(p)$ is increasing on $[0, 0.5]$, $U_{B1}(0) = a_0$ has to be 0.5. Also $U_{B1}(1) = U_B(1) > U_B(0) \geqslant 0.5$, therefore, (4.6) is established. □

Lemma 2. *If the smallest interval exists in the class $\mathscr{B}$, then for any two intervals $C_1(X)$ and $C_2(X)$ in $\mathscr{B}$, the intersection of $C_1(X)$ and $C_2(X)$ is also in $\mathscr{B}$.*

The proof is trivial and is omitted.

The outline of the proof of Theorem 6. Suppose the smallest interval exists when (4.6) is not true. Let

$$y_0 = \min\{y \in S_B : a_0 = U_{B1}(0) < L_{B1}(y)\}.$$

Due to (4.6),

$$y_0 \leqslant n-1.$$

There are two cases:

(I) $L_B(y_0)(= L_{B1}(y_0)) < 0.5$,
(II) $L_B(y_0)(= L_{B1}(y_0)) \geqslant 0.5$.

For case (I), we will construct two $1-\alpha$ confidence intervals, denoted by $[L_2(X), U_2(X)]$ and $[L_3(X), U_3(X)]$, in the class $\mathscr{B}$ and show that the intersection of these two intervals does not have a confidence level $1-\alpha$, which contradicts with Lemma 2. For case (II), we will have a similar result. Therefore the smallest interval in $\mathscr{B}$ does not exist. □

Lemma 3. *The coverage probability for the interval* $[L_{B1}(X), U_{B1}(X)]$, $f_{B1}(p)$, *is no less than* $1-\alpha$ *for* $p \in [0, U_{B1}(0))$.

Proof. The interval $[0, U_{B1}(0))$ is the union of disjoint intervals $\{I_i = [L_{B1}(i), L_{B1}(i+1))\}_{i=0}^{y_0-2}$ and $I_{y_0-1} = [L_{B1}(y_0-1), U_{B1}(0))$. On each I_i,

$$\begin{aligned} F_{B1}(P) &= \sum_{x=0}^{i} \operatorname{bin}(x; n, p) \geqslant F_B(i; n, L_{B1}(i+1)) \\ &= F_B(i; n, L_B(i+1)) = 1-\alpha. \end{aligned}$$

Therefore, $f_{B1}(p)$ is no less than $1-\alpha$ on $[0, U_{B1}(0))$. □

In the next two lemmas, we construct two intervals in the class $\mathscr{B}$ for case I.

Lemma 4. *Under the condition of Theorem* 6, *if*

$$L_B(y_0) < 0.5 \ (\textit{i.e.}, \ \textit{case}\ \mathrm{I}), \tag{A.5}$$

let

$$U_2(x) = \begin{cases} L_B(y_0) & \textit{if } x = 0, \\ 1 - L_B(y_0) & \textit{if } 1 \leqslant x \leqslant n - y_0, \\ U_B(x) & \textit{if } x \geqslant n - y_0 + 1. \end{cases} \tag{A.6}$$

and let

$$L_2(X) = 1 - U_2(X).$$

Then the interval $[L_2(X), U_2(X)]$ *belongs to* $\mathscr{B}$.

Proof. It is easy to see that $U_2(x)$ is nondecreasing in x. So the interval $[L_2, U_2]$ belongs to $\mathscr{B}$ if its coverage probability function, denoted by $f_2(p)$, is at least $1-\alpha$ on $[0, 0.5]$. The confidence limits of $[L_2(X), U_2(X)]$ on $[0, 0.5]$ are listed in a nondecreasing order as

$$L_2(0)(=L_B(0)), \ldots, L_2(y_0-1)(=L_B(y_0-1)), L_2(y_0) = \cdots = L_2(n-1) = U_2(0)$$

since $U_2(0) = L_B(y_0) < 0.5$.

(a) On $[0, U_{B1}(0))$, $f_2(p)$ is equal to $f_{B1}(p)$, and therefore no less than $1-\alpha$ by Lemma 3.
(b) On $[U_{B1}(0), U_2(0))$, note $L_2(y_0-1) = L_B(y_0-1) \leqslant U_{B1}(0)$ and $L_2(y_0) = L_B(y_0)$. So

$$f_2(p) = \sum_{x=0}^{y_0-1} \operatorname{bin}(x; n, p) \geqslant \sum_{x=0}^{y_0-1} \operatorname{bin}(x; n, U_2(0)) = 1-\alpha$$

due to $U_2(0) = L_B(y_0)$.
(c) At $p = U_2(0) = L_2(y_0) < 0.5$, since $0.5 < L_B(n)$ and $L_2(x) = L_2(y_0)$ for $y_0 \leqslant x < n$,

$$f_2(p) = \sum_{x=0}^{n-1} \operatorname{bin}(x; n, p) = F_B(n-1; n, p) > F_B(n-1; n, L_B(n)) = 1-\alpha.$$

(d) On $(U_2(0), 0.5]$,

$$f_2(p) = \sum_{x=1}^{n-1} \text{bin}(x; n, p) = g(p) \geqslant g(a_0) = 1 - \alpha,$$

where $g(p)$ is given in (4.2). Combining (a)–(d), $f_2(p) \geqslant 1 - \alpha$ on $[0, 0.5]$. $\square$

Lemma 5. *Under the condition of Theorem* 6 (*do not need* (A.5)), *let*

$$U_3(x) = \begin{cases} a_0 & \text{if } x = 0, \\ 1 - a_0 & \text{if } 1 \leqslant x \leqslant n - y_0, \\ U_B(x) & \text{if } x \geqslant n - y_0 + 1. \end{cases} \tag{A.7}$$

and let

$$L_3(X) = 1 - U_3(X).$$

Then the interval $[L_3(X), U_3(X)]$ *belongs to* $\mathscr{B}$.

Proof. The confidence limits of $[L_3(X), U_3(X)]$ on $[0, 0.5]$ are listed in a nondecreasing order as

$$L_3(0)(=L_B(0)), \ldots, L_3(y_0 - 1)(=L_B(y_0 - 1)), L_3(y_0) = \cdots = L_3(n - 1) = U_3(0),$$

since $U_3(0) = a_0 < 0.5$. On each interval $[L_3(i), L_3(i + 1))$ for $0 \leqslant i \leqslant y_0 - 2$, the coverage probability of $[L_3(X), U_3(X)]$, denoted by $f_3(p)$, is no less than $1 - \alpha$ as shown in Lemma 3 since $U_3(0) = a_0 \geqslant L_B(y_0 - 1)$; on $[L_3(y_0 - 1), L_3(y_0))$,

$$f_3(p) = \sum_{x=0}^{y_0-1} \text{bin}(x; n, p) \geqslant F_B(y_0 - 1; n, a_0) \geqslant F_B(y_0 - 1; n, L_B(y_0)) = 1 - \alpha$$

due to $a_0 < L_B(y_0)$; at $p = L_3(y_0) = a_0$,

$$f_3(a_0) = \sum_{x=0}^{n-1} \text{bin}(x; n, a_0) > g(a_0) = 1 - \alpha;$$

on $(U_3(0), 0.5]$,

$$f_3(p) = \sum_{x=1}^{n-1} \text{bin}(x; n, p) > g(a_0) = 1 - \alpha.$$

So we conclude that $[L_3(X), U_3(X)]$ belongs to $\mathscr{B}$. $\square$

The Proof of Theorem 6 for case (I). Let $[L_4(X), U_4(X)]$ be the intersection of two intervals $[L_2, U_2]$ and $[L_3, U_3]$. Then

$$U_4(x) = \begin{cases} a_0 & \text{if } x = 0, \\ 1 - L_B(y_0) & \text{if } 1 \leqslant x \leqslant n - y_0, \\ U_B(x) & \text{if } x \geqslant n - y_0 + 1. \end{cases} \tag{A.8}$$

Since (4.6) is not true, $U_4(X)=1-L_B(y_0)$ with a positive probability. Since $F_B(y_0-1; n, L_B(y_0))= 1-\alpha$ and $d=\text{bin}(0; n, L_B(y_0))$ is positive, one can choose a small positive Δp so that $p_0 = L_B(y_0)-\Delta p \in (a_0, L_B(y_0))$ and the coverage probability of $[L_4, U_4]$ at p_0,

$$\begin{aligned} f_4(p_0) &= \sum_{x=1}^{y_0-1} \text{bin}(x; n, p_0) = F_B(y_0-1; n, p_0) - \text{bin}(0; n, p_0) \\ &< 1-\alpha + d/3 - 2d/3 < 1-\alpha. \end{aligned}$$

Therefore, $[L_4, U_4]$ does not belongs to $\mathscr{B}$, which contradicts with Lemma 2. Therefore, we conclude that the smallest interval in $\mathscr{B}$ does not exist when (4.6) is not true for case I). □

In the next lemma, we construct one interval in the class $\mathscr{B}$ for case (II).

Lemma 6. *Under the condition of Theorem* 6, *if*

$$L_B(y_0) \geqslant 0.5 \ (\textit{i.e., case II}) \tag{A.9}$$

let

$$U_5(x) = \begin{cases} 0.5 & \textit{if } x \leqslant n-y_0, \\ U_B(x) & \textit{if } x \geqslant n-y_0+1. \end{cases} \tag{A.10}$$

and let

$$L_5(X) = 1 - U_5(X).$$

Then the interval $[L_5(X), U_5(X)]$ *belongs to* $\mathscr{B}$.

Proof. The confidence limits of $[L_5(X), U_5(X)]$ on $[0, 0.5]$ are listed in a nondecreasing order as

$$L_5(0)(=L_B(0)), \ldots, L_5(y_0-1)(=L_B(y_0-1)),$$
$$0.5 = L_5(y_0) = \cdots = L_5(n) = U_5(0) = \cdots = U_5(n-y_0).$$

On each interval $[L_5(i), L_5(i+1))$ for $0 \leqslant i \leqslant y_0-2$, the coverage probability of $[L_5(X), U_5(X)]$, denoted by $f_5(p)$, is no less than $1-\alpha$ as shown in Lemma 3 since $U_5(0)=0.5 > L_5(y_0-1)$; on $[L_5(y_0-1), 0.5)$,

$$f_5(p) = \sum_{x=0}^{y_0-1} \text{bin}(x; n, p) \geqslant F_B(y_0-1; n, 0.5) \geqslant F_B(y_0-1; n, L_B(y_0)) = 1-\alpha$$

due to (A.9); and at $p=0.5$,

$$f_5(p) = 1 > 1-\alpha.$$

So we conclude that $[L_5(X), U_5(X)]$ belongs to $\mathscr{B}$. □

The Proof of Theorem 6 for case (II). Let $[L_6(X), U_6(X)]$ be the intersection of two intervals $[L_5, U_5]$ and $[L_3, U_3]$. Then

$$U_6(x) = \begin{cases} a_0 & \text{if } x=0, \\ 0.5 & \text{if } 1 \leqslant x \leqslant n-y_0, \\ U_B(x) & \text{if } x \geqslant n-y_0+1. \end{cases} \tag{A.11}$$

Since (4.6) is not true, $U_6(X) = 0.5$ with a positive probability. Let $f_6(p)$ be the coverage probability of $[L_6, U_6]$. Since $a_0 < 0.5 = L_6(y_0)$, $g(a_0) = 1 - \alpha$, and $d = \sum_{x=y_0}^{n-1} \mathrm{bin}(x; n, a_0)$ is positive, one can choose a small positive Δp so that $p_0 = a_0 + \Delta p \in (a_0, 0.5)$ and

$$\begin{aligned} f_6(p_0) &= \sum_{x=1}^{y_0-1} \mathrm{bin}(x; n, p_0) \\ &= g(p_0) - \sum_{x=y_0}^{n-1} \mathrm{bin}(x; n, p_0) < 1 - \alpha + d/3 - 2d/3 < 1 - \alpha. \end{aligned}$$

Therefore, $[L_6, U_6]$ does not belong to $\mathscr{B}$, which contradicts with Lemma 2. Therefore, we conclude that the smallest interval in $\mathscr{B}$ does not exist when (4.6) is not true for case (II). Combining cases (I) and (II), we establish the nonexistence of the smallest interval in $\mathscr{B}$ when (4.6) is not true. $\square$

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