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ANALYSIS OF NEARLY SATURATED DESIGNS USING COMPOSITE VARIANCE ESTIMATORS

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SYNOPTIC ABSTRACT

Langsrud and Naes (1998) proposed forward-selection and backward-elimination strategies for the analysis of nearly saturated designs using composite variance estimators. Their variance estimators combine an estimator that is a function of the smaller sums of squares of the effect estimators (assuming effect sparsity) with an independent variance estimator based on the available error degrees of freedom. However, exact control of error rates for their stepwise methods remains an open problem. We investigate procedures that likewise use composite variance estimates but also provide exact control of error rates.

Keywords: composite estimator; effect sparsity; fractional factorial design; nearly saturated.

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1. INTRODUCTION.

A design is said to be saturated if there are exactly as many observations as (linear) parameters to estimate in the corresponding linear model. A saturated design provides no error degrees of freedom, so there is no mean squared error (*mse*) with which to estimate the error variance σ^2. However, if most of the parameters to be estimated are anticipated to be negligible—a condition known as effect sparsity—then the smaller parameter sums of squares can be used to form a surrogate for the *mse*, a quasi mean squared error (*qmse*). If a design is nearly saturated—namely, if the design has only a few more observations than parameters to estimate—then it is advantageous to combine these two estimators *qmse* and *mse* to obtain a more stable composite estimator of the error variance. In this paper, methods of analysis of nearly saturated designs are provided which use composite variance estimators and still provide control of error rates (Sections 2 and 4). Consideration is also given for how best to form the composite variance estimators (Section 3).

There has been great interest in recent years in problems concerning the analysis of saturated designs. Many methods of analysis have been proposed. Hamada and Balakrishnan (1998) provide an excellent review of these. Relatively few of the proposed methods are known to control error rates uniformly over the parameter space. See Kinateder, Voss, and Wang (2000b) for a review of such methods and a discussion of related open problems. Recently, Langsrud and Naes (1998) proposed some stepwise methods of analysis of saturated and nearly saturated designs under effect sparsity. They did not establish control of error rates for their proposed methods of analysis, but their paper is noteworthy for the following reason. For the analysis of a nearly saturated design under the assumption of effect sparsity, they apparently are the first to propose the use of a composite variance estimator of the type considered here.

Control of error rates in the analysis of non-orthogonal designs is a rather perplexing problem. Kinateder, Voss and Wang (2000a) provided one such method, which they illustrated in the analysis of a nearly-saturated non-orthogonal design. In the next section we illustrate the improvement obtained when the same data are re-analyzed using composite variance estimators.

Throughout this paper, observations are assumed to be independent and normally distributed with homogeneous variance σ^2, and effect estimators are ordinary least squares estimators. Thus, for any sum of squares SS obtained in analysis of variance, the ratio SS/σ^2 has a χ^2 distribution, either central or non-central.

Table 1: Plackett-Burman design for four factors with 12 runs.

Run	y	A	B	C	D	AB	AC	AD	BC	BD	CD
1	28.12	1	−1	1	−1	−1	1	−1	−1	1	−1
2	31.15	1	1	−1	1	1	−1	1	−1	1	−1
3	21.86	−1	1	1	−1	−1	−1	1	1	−1	−1
4	29.17	1	−1	1	1	−1	1	1	−1	−1	1
5	33.96	1	1	−1	1	1	−1	1	−1	1	−1
6	33.12	1	1	1	−1	1	1	−1	1	−1	−1
7	22.36	−1	1	1	1	−1	−1	−1	1	1	1
8	18.76	−1	−1	1	1	1	−1	−1	−1	−1	1
9	18.06	−1	−1	−1	1	1	1	−1	1	−1	−1
10	25.99	1	−1	−1	−1	−1	−1	−1	1	1	1
11	20.95	−1	1	−1	−1	−1	1	1	−1	−1	1
12	17.23	−1	−1	−1	−1	1	1	1	1	1	1

2. ANALYSIS OF A NEARLY-SATURATED NON-ORTHOGONAL DESIGN

In this section, we motivate the use of composite estimators with an example. Consider the data and experimental design shown in Table 1. The design contains 12 runs. The corresponding combinations of levels of the factors A, B, C and D were obtained taking four columns of the 12-run Plackett-Burman (1946) design. Kinateder, Voss and Wang (2000a) provided an analysis of these data using a model including main effects and two-factor interactions. Plackett-Burman designs are orthogonal main-effect plans, but the designs are non-orthogonal when two-factor interactions are included in the model.

The design is nearly saturated. There is one degree of freedom for error, so standard methods of analysis may be used. If a standard analysis of variance is performed, none of the effects are found to be statistically significantly nonzero. The most significant effect is the main effect of factor A, with an observed significance level of 0.0898—the estimate of the main effect of A (using regressors $\pm 1/2$) is 10.296, $mse = 3.948$, and the value of the t-statistic for testing for no main effect of A is $t = 7.041$. The individual 95% confidence interval for the main effect of A is (10.296 ± 18.576). The problem is that one error degree of freedom provides very little power.

Kinateder, Voss and Wang (2000a) provided an alternative analysis, utilizing methods for the analysis of saturated designs, treating the one error degree of freedom as if it corresponded to some treatment effect. They obtained a 95% confidence interval for the

Table 2: Plackett-Burman design for four factors with 12 runs.

Effect	Type I SS	Type III SS
B	56.637	35.269
C	3.050	3.336
D	3.193	1.669
AB	1.534	0.933
AC	0.847	0.537
AD	0.194	0.151
BC	50.009	0.009
BD	40.632	0.000
CD	37.060	0.057
A	195.700	195.700

main effect of A as follows. Consider the sequential sums of squares obtained by adding the main effects and two-factor interaction effects to the model one by one in the following predetermined order: first enter into the model the main effects of B, C and D, then the interaction effects AB, AC, AD, BC, BD, CD, and finally the main effect of A. These sums of squares, referred to as Type I Sums of Squares in the SAS software and shown in Table 2, are independent of one another and of the error sum of squares, $sse = 3.948$. Also, since the main effect of A is entered into the model last, $ssA = \hat{\theta}_A^2/c_A$ for $c_A = \mathrm{Var}(\hat{\theta}_A)/\sigma^2$. Given the values $ssA = 195.700$ and $\hat{\theta}_A = 10.296$ (obtained using the SAS software), it follows that $c_A = (10.296)^2/195.7 \approx 0.54167$. Let $qmse_A$ be the mean of the six smallest sums of squares from the "other" 10 sums of squares—namely, from sse and the 9 sequential sums of squares excluding ssA. Then $qmse_A \approx 2.128$. Finally, let $s_A^2(\hat{\theta}_A) = 0.54167 \times QMSE_A$, analogous to a squared standard error but using $QMSE_A$ instead of MSE.

Following Kinateder, Voss and Wang (2000a), the distribution of

$$Q_A^2 = \frac{(\hat{\theta}_A - \theta_A)^2}{s_A^2(\hat{\theta}_A)}$$

is stochastically largest under the null distribution—namely, when all the effects θ_i are zero; (this follows from the Stochastic Ordering Lemma stated in Section 4). Letting $q_{0.05}^2$ denote the upper-0.05 quantile of the null distribution of Q_A^2, it follows that

$$P_{\boldsymbol{\theta}}\left(\frac{\left(\hat{\theta}_A - \theta_A\right)^2}{s_A^2(\hat{\theta}_A)} \le q_{0.05}^2\right) \ge 0.95$$

for all parameter configurations $\boldsymbol{\theta} = (\theta_A, \theta_B, \ldots, \theta_{CD})$, and the equality holds at the null case. Hence, an individual 95% confidence interval for θ_A is

$$\hat{\theta}_A \pm q_{0.05} s_A(\hat{\theta}_A)\,.$$

By simulation, one can show that $q_{0.05} \approx 5.09$. The resulting 95% confidence interval for θ_A is

$$(10.296 \pm 5.463)\,.$$

While this is much tighter than the confidence interval (10.296 ± 18.576) obtained using the single error degree of freedom and traditional methods, one can do even better by recognizing the sum of squares for error as such. Specifically, consider quantities of the form

$$R_A^2 = \frac{(\hat{\theta}_A - \theta_A)^2}{c_A[a(QSSE_A) + b(SSE)]}\,,$$

where $a > 0$ and $b > 0$ are fixed in advance, $c_A = \mathrm{Var}(\hat{\theta}_A)/\sigma^2$, sse is the error sum of squares with one degree of freedom, and $qsse_A$ is the sum of the five smallest Type I sums of squares excluding ssA. While the denominator of R_A^2 is formed from six sums of squares other than ssA, as was the case for Q_A^2, now the error sum of squares is being recognized as such.

The distribution of R_A^2 is stochastically largest when all effects are zero. Letting $r_{0.05}^2$ denote the upper-0.05 quantile of the null distribution of R_A^2, it follows that

$$P_{\boldsymbol{\theta}}\left(\frac{\left(\hat{\theta}_A - \theta_A\right)^2}{c_A[a(QSSE_A) + b(SSE)]} \le r_{0.05}^2\right) \ge 0.95 \qquad (1)$$

for all parameter configurations $\boldsymbol{\theta}$, and the equality holds at the null case. Hence, an individual 95% confidence interval for θ_A is

$$\hat{\theta}_A \pm r_{0.05}\sqrt{c_A[a(qsse_A) + b(sse)]}\,. \qquad (2)$$

If $a = 3$ and $b = 1$ (values to be explained in the next section), with $c_A = 0.54167$ as before, $sse = 3.948$, $qsse = 8.818$ computed from 5 sums of squares, and $r_{0.05} = 1.19$ (obtained by simulation of the null distribution), the resulting 95% confidence interval for θ_A is

$$(10.296 \pm 4.829)\,.$$

This interval is only 88% as wide as the interval (10.296 ± 5.463) obtained treating sse as another effect sum of squares, demonstrating the potential gain in using a composite variance estimator.

More generally, let $r^2_\alpha(a, b, k, n, \nu)$ denote the upper-α percentile of the null distribution of R^2_A, where $QMSE_A$ is the mean of the n smallest of $k-1$ sums of squares excluding SSA, MSE has ν degrees of freedom, and $c_A = \mathrm{Var}(\hat{\theta}_A/\sigma^2)$.

Theorem 3 *The interval* $(\hat{\theta}_A \pm r_\alpha(a, b, k, n, \nu)\sqrt{c_A[a(qsse_A) + b(sse)]})$ *is an individual* $100(1-\alpha)\%$ *confidence interval for* θ_A. *The confidence interval is exact for some parameter configurations and conservative for others (see Kinateder, Voss and Wang, 2000a).*

Proof. The theorem follows from equation (1) generalized for probability $1-\alpha$. □

This generalizes the result of Kinateder, Voss and Wang (2000a), which corresponds to the case $a = 1$ and $b = 0$.

Remarks Non-orthogonality complicates the analysis. As already noted, the results depend on a predetermined order of effects to obtain Type I sums of squares, and the effect of interest must be added last. Also, in view of the Type I and Type III sums of squares for our example given in Table 2, the Type I sums of squares corresponding to the BC, BD and CD interactions are clearly influenced by the large estimated main effect of A. See Kinateder, Voss and Wang (2000a) for related discussion. (The Type III sum of squares of an effect is the reduction in error sum of squares obtained by adding the effect to the model last.)

The choice of n, the number of terms to pool into $qsse$, depends on the amount of effect sparsity anticipated. To increase power, one should choose n as large as possible but not to exceed the number of negligible effects.

3. CHOICE OF WEIGHTS FOR COMPOSITE ESTIMATORS.

In the example of the previous section, we used the composite error variance estimator

$$a(QSSE) + b(SSE)$$

with weights $a = 3$ and $b = 1$, deferring till now explanation of this choice of weights.

Denote the mean and variance of $QSSE/\sigma^2$ under the null case by μ_q and σ^2_q, respectively, and denote the number of error degrees of freedom for SSE by ν. The random variables $QSSE$ and SSE are independent. Minimizing the variance of the composite variance estimator

$$a(QSSE) + b(SSE)$$

subject to the constraint that the estimator be unbiased for the error variance σ^2, one obtains the following result.

Lemma 4 *For composite variance estimators of the form*

$$\hat{\sigma}^2 = a(QSSE) + b(SSE),$$

the minimum variance unbiased estimator (under the null distribution) is obtained when

$$a = 2\mu_q/[2\mu_q^2 + \nu\sigma_q^2]$$

and

$$b = \sigma_q^2/[2\mu_q^2 + \nu\sigma_q^2],$$

in which case

$$a/b = 2\mu_q/\sigma_q^2.$$

In practice, the confidence interval in (2) only depends on the ratio a/b—namely, use of the weights $b = 1$ and $a = 2\mu_q/\sigma^2_q$ for the composite variance estimator is equivalent to use of the weights a and b as identified in Lemma 4, since the constant $1/b$ can be factored out of the denominator estimator and incorporated into the constant r_α. For the case illustrated previously, $\mu_q \approx 1.203$ and $\sigma^2 \approx 0.811$ were estimated via simulation under the null distribution, yielding the value $a = 2.966 \approx 3$.

The composite estimator identified in Lemma 4, while a MVUE under the null distribution, is not uniformly best. Furthermore, one would hope that the effects are <u>not</u> all zero—otherwise, one wouldn't use a design and analysis predicated on effect sparsity. Anticipating that the effects are not all zero, one should give more weight to SSE than would be appropriate in the null case. For example, if exactly five of the effects (excluding the main effect of A) are zero and the other five are infinitely large, then $QSSE/\sigma^2 \sim \chi^2(5)$ has mean five and variance 10. Then from Lemma 4, the best estimator has $a = 1$ if $b = 1$.

We refer to the estimator with $a = b = 1$ as the <u>pooled</u> estimator, since it corresponds to adding up or pooling together the error sums of squares and the smallest effect sums of squares. The pooled estimator is convenient to use if the design is orthogonal, since then it is merely the 'error sum of squares' obtained by removing from the model those effects with smallest sums of squares.

In general, the best composite estimator—or equivalently, the best value of the ratio a/b—depends on the unknown parameter configuration θ, as well as on k and n. However in practice θ is unknown, so choice of the best composite estimator is not possible. Unless the degree of effect sparsity is severely over-estimated, the best composite estimator will be somewhere between the pooled estimator and the MVUE under the null distribution as given in Lemma 4. To investigate this issue further, we ran simulations to compare

Table 3: Expected confidence interval lengths

	ν							
Estimator	1	2	3	4	5	6	7	8
sse	10.07	3.66	2.83	2.55	2.38	2.29	2.27	2.19
pooled	2.28	2.27	2.23	2.18	2.16	2.13	2.13	2.11
MVUE	2.23	2.22	2.18	2.12	2.11	2.10	2.08	2.08
qsse	2.31	2.30	2.29	2.29	2.31	2.35	2.33	2.33

the performance of four estimators for saturated designs with $k = 15$ effects, forming *qsse* from the $n = 8$ smallest of the $k - 1 = 14$ "other" sums of squares, using $\nu = 1, 2, \ldots, 8$ degrees of freedom for *sse*. The four estimators considered were: *sse* (for which $a/b = 0$), the pooled estimator ($a/b = 1$), the MVUE ($a/b = 3$), and *qsse* ($a/b = \infty$). (For $k = 15$ and $n = 8$, $\mu_q \approx 1.855$ and $\sigma^2 \approx 1.255$ were estimated via simulation under the null distribution, yielding the value $a = 2.956 \approx 3$, coincidentally matching the value in the previously considered case.)

Expected lengths of individual 95% confidence intervals were computed under the null distribution. The results are shown in Table 3.

Expected confidence lengths for the pooled estimator are consistently nearly as short as those for the MVUE, and both range from modestly to substantially better than either *qsse* or *sse*. In choosing between the pooled estimator and the MVUE, the pooled estimator enjoys the virtue of simplicity, whereas the MVUE provides a slight gain in performance at the expense of needing to determine the ratio a/b given the number of effects k and the number of terms n included in *qsse*.

4. ANALYSIS OF NEARLY SATURATED DESIGNS USING COMPOSITE ESTIMATORS.

In the previous section we illustrated the use and benefit of composite variance estimators for a nearly saturated, non-orthogonal design. In this section we provide the methods of analysis known to control error rates when using composite variance estimators in the analysis of nearly saturated factorial designs. All but one of the results follows from the following lemma of Voss (1999), which follows from similar results of Alam and Rizvi (1966) and Mahamunulu (1967) used in the ranking and selection literature for identifying least favorable configurations.

Lemma 5 *(Stochastic Ordering Lemma) Let $F_{i\theta_i}(x)$, with real parameter θ_i, be a stochastically increasing family of distribution functions on the real line, for $i = 1, 2, \ldots, k$. Let $X_1, X_2, \ldots, X_k$ be independent random variables, where the distribution function of X_i is $F_{i\theta_i}(x_i)$. For any fixed i, $1 \le i \le k$, if the statistic $t = t(x_1, x_2, \ldots, x_k)$ is a non-increasing function of x_i when all x_j for $j \ne i$ are held fixed, then the distribution of $T = t(X_1, X_2, \ldots, X_k)$ is stochastically decreasing in θ_i.*

4.1 Individual tests: orthogonal designs

Berk and Picard (1991) provided individual tests of effects for the analysis of orthogonal saturated designs. The same methodology controls error rates if an orthogonal design is nearly saturated and a composite variance estimator is used.

Consider analysis of an experiment for which there are k independent effect sums of squares, each with one degree of freedom, plus ν error degrees of freedom from which SSE is computed. Let

$$SS_{(1)} < SS_{(2)} < \cdots < SS_{(k)}$$

denote the order statistics of the effect sums of squares. For testing the null hypothesis $H_0 : \theta_i = 0$ that the ith effect is zero, consider the test statistic

$$R_i = SS_i / [a(QSSE) + b(SSE)], \tag{6}$$

where $QSSE = \sum_{j=1}^{n} SS_{(j)}$ is the sum of the n smallest of k effect sums of squares, for pre-specified integer n. Let $r_\alpha(a, b, k, n, \nu)$ denote the upper-α percentile of the null distribution of R_i.

Theorem 7 *A size-α test of $H_0 : \theta_i = 0$ is to reject H_0 if $R_i > r_\alpha(a, b, k, n, \nu)$.*

Proof. The proof follows from the Stochastic Ordering Lemma. □

This generalizes the result of Berk and Picard (1991), which corresponds to the case $a = 1$ and $b = 0$.

4.2 Simultaneous tests: orthogonal designs

Voss (1988) provided simultaneous step-down tests of the k effects θ_i $(i = 1, 2, \ldots, k)$ for the analysis of orthogonal saturated designs. The same methodology controls error rates if an orthogonal design is nearly saturated and a composite variance estimator is used.

Consider analysis of an experiment for which there are k independent effect sums of squares, each with one degree of freedom, plus ν error degrees of freedom from which SSE is computed. As before, let

$$SS_{(1)} < SS_{(2)} < \cdots < SS_{(k)}$$

denote the order statistics of the effect sums of squares. For testing the null hypotheses $H_0 : \theta_i = 0$ for $i = 1, 2, \ldots, k$, consider the test statistics

$$R_{(i)} = SS_{(i)}/D\,, \tag{8}$$

where $D = a(QSSE) + b(SSE)$ and $QSSE = \sum_{j=1}^{n} SS_{(j)}$ is the sum of the n smallest of k effect sums of squares for pre-specified integer n. Let $r_\alpha(a, b, k, n, \nu)$ denote the upper-α percentile of the null distribution of

$$\max_{i \le k} SS_i/D\,.$$

The step-down testing procedure is as follows. Let $\theta_{(i)}$ denote the parameter corresponding to the ith smallest sum of squares, $ss_{(i)}$. If $ss_{(k)}/d > r_\alpha(a, b, k, n, \nu)$, then assert $\theta_{(k)} \neq 0$ and continue; otherwise stop. If $ss_{(k-1)}/d > r_\alpha(a, b, k-1, n, \nu)$, then assert $\theta_{(k-1)} \neq 0$ and continue; otherwise stop. Continue in this fashion, asserting $\theta_{(i)} \neq 0$ for each i such that $ss_{(j)}/d > r_\alpha(a, b, j, n, \nu)$ for all $j \ge i$.

Theorem 9 *This step-down testing procedure is of family-wise size α.*

Proof. The proof follows from the Stochastic Ordering Lemma, this being a closed test as considered by Marcus, Peritz and Gabriel (1976). □

This generalizes the result of Voss (1988), which corresponds to the case $a = 1$ and $b = 0$.

Remark Iterative application of step-down tests is known to provide sharper critical values and hence more powerful tests. Such methods have been recommended by Zahn (1975ab), Venter and Steel (1998), Langsrud and Naes (1998), and Ye, Hamada and Wu (2001), and that they are more powerful was noted by Voss (1988). However, such iteratively-applied step-down tests are not closed tests, and it remains an open problem to show that such procedures control the experimentwise error rate under all parameter configurations. This and related problems are discussed by Kinateder, Voss and Wang (2000b).

4.3 Individual confidence intervals: orthogonal designs

Voss (1999) provided individual confidence intervals for effects in the analysis of orthogonal saturated designs. The same methodology controls error rates if an orthogonal design is nearly saturated and a composite variance estimator is used.

Consider analysis of an experiment for which there are k independent effect sums of squares, each with one degree of freedom, plus ν error degrees of freedom from which SSE is computed. Without loss of generality, consider constructing a confidence interval for θ_k, the kth effect. Let

$$SS_{(1:k-1)} < SS_{(2:k-1)} < \cdots < SS_{(k-1:k-1)}$$

denote the order statistics of the "other" $k-1$ effect sums of squares—namely, excluding the sum of squares for the kth effect. The confidence interval can be obtained from the following pivotal quantity.

$$Q_k^2 = (\hat{\theta}_k - \theta_k)^2/[c_k(a(QSSE_k) + b(SSE))]\,, \tag{10}$$

where $c_k = \mathrm{Var}(\hat{\theta}_k)/\sigma^2$ and $QSSE_k = \sum_{j=1}^{n} SS_{(j:k-1)}$ is the sum of the n smallest of $k-1$ "other" effect sums of squares for pre-specified integer n, $(n \le k-1)$. Let $q_\alpha^2(a, b, k, n, \nu)$ denote the upper-α percentile of the null distribution of Q_k^2.

Theorem 11 *The interval*

$$\left(\hat{\theta}_k \pm q_\alpha(a, b, k, n, \nu)\sqrt{c_k[a(qsse_k) + b(sse)]}\right)$$

is an individual $100(1-\alpha)\%$ *confidence interval for θ_k. The confidence level is exact if all effects are zero. If any effects other than θ_k are nonzero, then the confidence interval is conservative.*

Proof. The proof follows from the Stochastic Ordering Lemma. □

This generalizes the result of Voss (1999), which corresponds to the case $a = 1$ and $b = 0$.

4.4 Simultaneous confidence intervals: orthogonal designs

All of the results presented previously in this section follow from the Stochastic Ordering Lemma. The next result presented does not.

Voss and Wang (1999) provided simultaneous confidence intervals for effects in the analysis of orthogonal saturated designs. The same methodology controls error rates if an orthogonal design is nearly saturated and a composite variance estimator is used.

Consider analysis of an experiment for which there are k independent effect sums of squares, each with one degree of freedom, plus ν error degrees of freedom from which SSE is computed. Without loss of generality, consider constructing a confidence interval for each of the effects θ_i, $i = 1, 2, \ldots, k$. The confidence intervals can be obtained as follows. Let

$$Q_i^2 = (\hat{\theta}_i - \theta_i)^2/[c_i(a(QSSE_i) + b(SSE))]\,, \tag{12}$$

where $c_i = \mathrm{Var}(\hat{\theta}_i)/\sigma^2$ and $QSSE_i$ is the sum of the n smallest of $k-1$ "other" effect sums of squares (excluding SS_i) for pre-specified integer n, $(n \le k-1)$. Let $M^2 = \max_{1\le i\le k} Q_i^2$, and let $m_\alpha^2(a,b,k,n,\nu)$ denote the upper-α percentile of the null distribution of M^2.

Theorem 13 *The intervals*

$$\left(\hat{\theta}_i \pm m_\alpha(a,b,k,n,\nu)\sqrt{c_i[a(qsse_i) + b(sse)]}\right)$$

$(i = 1,2,\ldots,k)$ are simultaneous $100(1-\alpha)\%$ confidence intervals for $\theta_1, \theta_2, \ldots, \theta_k$, respectively. The confidence level is exact if all effects are zero. If any effects are nonzero, then the confidence intervals are conservative.

Proof. The same as in Voss and Wang (1999). □

This generalizes the result of Voss and Wang (1999), which corresponds to the case $a = 1$ and $b = 0$.

5. CLOSING REMARKS.

Methods of analysis of nearly saturated designs have been presented which use composite variance estimators and control error rates over all parameter configurations. The five theorems concerning control of error rates directly generalize corresponding results for the analysis of saturated designs. The theorems require pre-specification of the number n of terms to be included in *qsse*. To enhance power, n should be set as large as possible without forcing inclusion of sums of squares of non-negligible effects.

The individual tests of Theorem 7, stated for the analysis of orthogonal designs, also apply for the analysis of non-orthogonal designs. The adaptations required are analogous to those illustrated in Section 2 for individual confidence intervals.

Use of the composite variance estimators provides greater power than use of only *sse* or its surrogate *qsse*. Of the composite estimators, the pooled estimator *qsse*+*sse* is convenient to use and nearly best. Sample code for computing critical values using the pooled estimator is provided in the appendix.

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APPENDIX

To facilitate implementation of our procedures, Table 4 contains a SAS simulation program for computing critical values $r_\alpha(1,1,k,n,\nu)$ as required by equation (2) for the case of: the pooled estimator *qsse* + *sse*, individual $100(1-\alpha)\%$ confidence intervals for $\alpha = 0.10, 0.05, 0.01$, $k = 15$ effects, $n = 8$ terms for *qsse*, and $\nu = 5$ (i.e. "nu"=5) error degrees of freedom, based on 9999 runs. These parameters are easily changed to suit the user's needs.

Table 4: SAS program for computing critical values

```
* cvalues.sas;
* compute critical values for a single CI, given k=15 effects,
* n=8 sums of squares for qsse, and nu=5 d.f. for sse;
options ls=75 nocenter;
;
data sim;
 * initialize parameters;
 array seed(15);
 array ss(15) ss1-ss15;
 k=15; * number of effects;
 nu=5; * number of error d.f.;
 n=8; * number of terms in qsse;
 nsims=9999; * number of samples for simulation;
 keep nsims r k n nu;
 * initialize seeds;
 do i=1 to k; seed(i) = 2*ranbin(5833157,10000,.5) + 1; end;
 * generate nsims samples;
 do m=1 to nsims;
  call rannor(seed(1),z); ss(1)=z*z;
  do i=2 to k; call rannor(seed(i),z); ss(i)=z*z; end;
  * sort the first k-1 effects;
  do v=2 to (k-1);
   * sort the first v effects;
   do i=v to 2 by -1;
    if ss(i-1) > ss(i) then
     do temp=ss(i-1); ss(i-1)=ss(i); ss(i)=temp; end;
   end; * i loop;
  end; * end v loop;
   ***** compute qsse from the n smallest effects;
   qsse=0; do i=1 to n; qsse=qsse+ss(i); end;
   ***** compute sse;
  sse=0; do i=1 to nu; call rannor(seed(i),z); sse=sse+z*z; end;
  * compute critical values;
  r2 = ss(k)/(qsse+sse);
  r=sqrt(r2);
  output;
 end; * end m;
;
***** COMPUTE CRITICAL VALUES;
data cvalues; set sim;
proc sort; by r;
data cvalues; set cvalues; keep r alpha k n nu;
   alpha = (nsims+1-_n_)/(nsims+1); * alpha = 1 - (_n_/(nsims+1));
   if alpha=0.10 or alpha=0.05 or alpha=0.01;
proc print;
```


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function. Second-order appr
justifications to some of the
given. 'Improved' estimator
estimators is established.

Key Words and Phrases: g
improved estimators; second

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