SHARP REGULARITY RESULTS ON SECOND DERIVATIVES OF SOLUTIONS TO THE MONGE-AMPÈRE EQUATION WITH VMO TYPE DATA

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Abstract. We establish interior estimates for L^p norms, Orlicz norms and mean oscillation of second derivatives of solutions to the Monge-Ampère equation $\det D^2u = f(x)$ with zero boundary value, where $f(x)$ is strictly positive, bounded, and satisfies a VMO type condition. These estimates develop the regularity theory of the Monge-Ampère equation in VMO type spaces. Our Orlicz estimates also sharpen Caffarelli's celebrated $W^{2,p}$ estimates.

§1. Introduction

This paper is concerned with interior regularity of weak solutions to the Monge-Ampère equation

$$\det D^2u = f(x) > 0, \tag{1-1}$$

where $x \in \Omega$, Ω is a bounded convex domain in $\mathbb{R}^n$, and $0 < \lambda \le f \le \Lambda$ in Ω for some constants λ, Λ.

The regularity of solutions for the Monge-Ampère equation has been extensively studied by many authors. For instance, see [P1], [Ch-Y], [C-N-S], [Kr1-3], [C2-3], [W], [H], [T-W], and references therein.

If $f \in C^3$, $u|_{\partial\Omega} = g$ with $g \in C^4$, and Ω is a C^∞ strictly convex domain, i.e., given data are very smooth, [C-N-S] established interior and boundary $C^{2,\alpha}$ estimates. Recently, [T-W] established sharp global $C^{2,\alpha}$ estimates under the assumptions that $f \in C^\alpha(\overline{\Omega})$, Ω is a C^3 strictly convex domain, and $g \in C^3$. On the other hand, if boundary values are not smooth enough, even though f is a constant, Pogorelov type examples show solutions of (1-1) may contain a segment and may be in only $C^{1,1-2/n}$. This problem of strict convexity presented a great obstacle for the study of local properties of weak solutions. Caffarelli pioneered studies in this direction by overcoming the difficulty of strict convexity in [C2].

2000 *Mathematics Subject Classification.* Primary 35J60; Secondary 35B65.

Key words and phrases. Monge-Ampère equation, regularity, section, Orlicz norms, $W^{2,p}$ estimates, mean oscillation, VMO, VMO^ψ.

* The author was partially supported by NSF grants DMS-0201599 and DMS-0502045.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

By [C2], if the boundary value g of a solution u to (1-1) is $C^{1,\alpha}$ and $\partial\Omega$ is $C^{1,\alpha}$ with $\alpha > 1 - 2/n$, then u is strictly convex. Since u is affine on the boundary of a cross section of u, to study interior estimates, one can assume the boundary value of u is zero through transformations.

In the case that f is continuous, interior regularity of second derivatives of solution u to (1-1) with $u|_{\partial\Omega} = 0$ was well investigated. If $f \in C^\alpha$ $(0 < \alpha < 1)$, then $D^2u \in C^\alpha$ by the Caffarelli-Schauder estimates in [C3]. If f is Dini continuous, [W] (also [Ji-W]) proved that D^2u is bounded and continuous. If f is non-Dini continuous with modulus of continuity φ satisfying $\varphi(r) = o\left(\frac{1}{\log(1/r)}\right)$, [H] proved that $D^2u \in \text{VMO}$, where VMO is the subspace of BMO consisting of functions with vanishing mean oscillation, and furthermore, the mean oscillation of D^2u has the modulus $\leq \log^{-(1-\varepsilon)}(1/r)$ for any $0 < \varepsilon < 1$. If f is continuous, then $D^2u \in L^p$ for any $1 \leq p < \infty$, by the celebrated interior $W^{2,p}$ estimates in [C3].

In light of the regularity theory for fully nonlinear uniformly elliptic equations in [C1], [C-H], one may ask what regularity the second derivatives of solutions to (1-1) can have when f is discontinuous. If f is only strictly positive and bounded, an example in [W] shows that u might not be in $W^{2,p}$. This is quite unlike the $W^{2,p}$ theory of fully nonlinear uniformly elliptic equations. We note that f in this example in [W] is not in VMO space, since it has jump discontinuity along some smooth convex curves and as a result its mean oscillation over a cube centered at the curves is not going to zero as the diameter of the cube approaches to zero.

Our purpose in this paper is to consider L^p norms, Orlicz norms, and mean oscillation of second derivatives of solutions to (1-1) when $f(x)$ satisfies some VMO type condition and may be discontinuous.

Before stating the main results, we recall spaces $\text{VMO}(\Omega)$ and $\text{BMO}_\psi(\Omega)$ (redenoted by $\text{VMO}^\psi(\Omega)$ if $\psi(0) = 0$), and introduce spaces $\text{VMO}_{loc}(\Omega, u)$ and $\text{VMO}^\psi_{loc}(\Omega, u)$.

Let ψ be a nondecreasing continuous function on $[0, \infty)$ such that $\psi(t) > 0$ for $t > 0$ and $t/\psi(t)$ is almost increasing which means $t/\psi(t) \leq Ks/\psi(s)$ for $0 < t < s$. For $g(x) \in L^1(\Omega)$ and $A \subset \Omega$, the mean oscillation of $g(x)$ over A is defined by

$$\text{mosc}_A g = ⨍_A |g(x) - g_A| \, dx,$$

Where $g_A = ⨍_A g \, dx$ denotes the average of g over A. Let $B_r(x_0)$ be the ball centered at x_0 with radius r and for simplicity let $B_r = B_r(0)$.

A function $g(x) \in L^1(\Omega)$ belongs to $\text{BMO}_\psi(\Omega)$ if there exists a constant C such that

$$\text{mosc}_{B_r(x_0)\cap\Omega} g \leq C\psi(r),$$

for all $x_0 \in \Omega$, $0 < r \leq d = diam(\Omega)$, where $diam(\Omega)$ is the diameter of Ω. We recall that $g(x) \in \text{VMO}(\Omega)$ if and only if $\text{mosc}_{B_r(x_0)\cap\Omega} g$ converges to 0 uniformly in $x_0 \in \Omega$ as $r \longrightarrow 0$. For further properties of $\text{BMO}_\psi(\Omega)$, see [C-H] and references therein. If $\psi \equiv 1$, $\text{BMO}_\psi(\Omega)$ is the usual $\text{BMO}(\Omega)$ space. Obviously, if $f \in \text{BMO}_\psi(\Omega)$ with $\psi(0) = 0$, then f has vanishing mean oscillation of modulus $C\psi$. For this sake, in this paper, we set $\text{VMO}^\psi(\Omega) = \text{BMO}_\psi(\Omega)$ if $\psi(0) = 0$.

To introduce $\mathrm{VMO}_{loc}(\Omega, u)$ and $\mathrm{VMO}^{\psi}_{loc}(\Omega, u)$, we recall the notion of (cross) sections of a convex solution.

Let $u(x)$ be a convex solution of (1-1) in Ω with f satisfying that $\lambda \le f \le \Lambda$ in Ω, and $u = 0$ on $\partial\Omega$. For $x_0 \in \Omega$, the section $S_h(x_0) = S_h(x_0, u)$ for $h > 0$ is defined by

$$S_h(x_0) = \{x \in \Omega : u(x) < \ell_{x_0}(x) + h\},$$

where $\ell_{x_0}(x) = u(x_0) + Du(x_0)(x - x_0)$ is the supporting hyperplane of u at x_0. By [C2], for any $\Omega' \subset\subset \Omega$, there exists h_0 such that

$$S_h(x_0) \subset\subset \Omega \qquad \text{for } h \le h_0,\ x_0 \in \Omega'.$$

Moreover as $h \to 0$, $diam(S_h(x_0)) \to 0$ uniformly in $x_0 \in \Omega'$.

Given a function $g(x) \in L^1(\Omega)$, we say that $g \in \mathrm{VMO}_{loc}(\Omega, u)$ if for any $\Omega' \subset\subset \Omega$

$$Q_g(r, \Omega') = \sup_{\substack{x_0 \in \Omega' \\ diam(S_h(x_0,u)) \le r}} \mathrm{mosc}_{S_h(x_0,u)} g \longrightarrow 0, \qquad \text{as } r \to 0.$$

A function $g \in L^n(\Omega)$ is said to be in $\mathrm{VMO}^{\psi}_{loc}(\Omega, u)$ where $\psi(0) = 0$ if for any $\Omega' \subset\subset \Omega$, there exists $C = C(\Omega')$ such that for all $x_0 \in \Omega'$ and $S_h(x_0, u) \subset\subset \Omega$

$$\mathrm{mosc}^{(n)}_{S_h(x_0,u)} g \le C\psi(diam(S_h(x_0, u))),$$

where $\mathrm{mosc}^{(n)}_A g = \big(⨍_A |g - g_A|^n \, dx \big)^{1/n}$. Denote by $[f]_{\mathrm{VMO}^{\psi}(\Omega',u)}$ the smallest of all such constants $C(\Omega')$.

The following theorems are the main results of this paper.

Theorem A. *Let u be a convex weak solution to* (1-1) *in Ω and $u = 0$ on $\partial\Omega$. Let $0 < \alpha_n < 1$ be a dimensional constant and $B_{\alpha_n} \subset \Omega \subset B_1$ a convex domain. Assume that there are constants λ, $\Lambda > 0$ such that $\lambda \le f(x) \le \Lambda$ in Ω. We have:*

(i) *If $f \in VMO_{loc}(\Omega, u)$, then $D^2u \in L^p_{loc}(\Omega)$ for any $1 \le p < \infty$.*

(ii) *If $f \in VMO^{\psi}_{loc}(\Omega, u)$ with $\psi(0) = 0$ and $\frac{\log(1/\psi(t))}{\log(1/t)}$ is almost increasing, then $\Psi(|D^2u|) \in L^1_{loc}(\Omega)$, where $\Psi(t) = t^{-C\log\psi(\frac{1}{t})}$ for $t \ge 1$ and $C > 0$ is a constant dependent only on λ, Λ, n and ψ.*

Theorem B. *Let u and Ω satisfy the assumptions in Theorem A. Suppose that $\lambda \le f(x) \le \Lambda$ in Ω. Assume that $t^{\gamma}/\psi(t)$ is almost increasing for some $0 < \gamma < 1$. Then*

(i) *If $f \in VMO^{\psi}_{loc}(\Omega, u)$ with $\psi(t) = o(\log^{-1}(4/t))$, then $D^2u \in VMO^{\log^{\varepsilon-1}(4/t)}_{loc}(\Omega)$ for any $0 < \varepsilon < 1$; if in addition $\frac{\psi(t)}{\log^{-1}(4/t)}$ is increasing, then $D^2u \in VMO^{\varphi_\tau}(\Omega')$ for any $\Omega' \subset\subset \Omega$ and small $\tau > 0$, where $\varphi_\tau(t) = \psi(t)e^{C\int_t^2 \frac{\psi(\tau r)}{r}}$ and C is a constant dependent only on n, λ, Λ, γ, ψ, $[f]_{\mathrm{VMO}^{\psi}(\Omega',u)}$ but not τ.*

(ii) *If $f \in VMO^{\psi}_{loc}(\Omega, u)$ and the Dini condition $\int_0^1 \frac{\psi(t)}{t} < \infty$ holds, then $D^2u \in VMO^{\psi}_{loc}(\Omega)$ and hence $D^2u \in C(\Omega)$.*

We remark that the space of all continuous functions on $\overline{\Omega}$ with the modulus of continuity of order ψ is generally a proper subspace of $\mathrm{VMO}^{\psi}_{loc}(\Omega, u)$. See Example 2.3, Remark 2.4, Example 7.2, Remark 7.3 for examples of functions in $\mathrm{VMO}^{\psi}_{loc}(\Omega, u)$. We also recall that if $f \in \mathrm{VMO}^{\psi}(\Omega)$ with $\int_0^1 \frac{\psi(t)}{t} < \infty$ then $|f(x) - f(y)| \le C \int_0^{|x-y|} \frac{\psi(t)}{t}$. Therefore, Theorem A sharpens Caffarelli's $W^{2,p}$ estimates in [C3], and Theorem B sharpens the estimates of second derivatives in [W], [Ji-W], [H] in the case that $\psi(t)$ converges to zero slower than some t^γ $(0 < \gamma < 1)$. The Orlicz estimates in (ii) of Theorem A are also new in the case of continuous $f(x)$. More explanations are given in sections 5-7.

The organization of this paper is as follows. In §2, we give some preliminary on sections and examples of functions satisfying or dissatisfying VMO type conditions. In §3, a comparison principle in terms of L^n norm is given. In §4, we discuss the eccentricity of cross sections if $f(x)$ satisfies VMO type conditions. In §5, $C^{1,\alpha}$ and $C^{2,\alpha}$ estimates are derived from the estimates of the eccentricity. In §6, we establish the $W^{2,p}$ and Orlicz estimates in Theorem A. Finally, Theorem B is proved in §7.

§2. Preliminary

We first give a variant of [Gu-Hu, Theorem 2.3] about the sizes of sections. For more properties of sections, see [Gu] or [Gu-Hu].

Lemma 2.1. *Let u be a convex solution to (1-1) in Ω and $u = 0$ on $\partial\Omega$. Assume that $B_{\alpha_n} \subset \Omega \subset B_1$ and $\lambda \le f(x) \le \Lambda$. Then for any $\Omega' \subset\subset \Omega$, there exist constants h_0, C_1, C_2, and q such that for $x_0 \in \Omega'$ and $0 < h \le h_0$*

$$B_{C_1 h}(x_0) \subset S_h(x_0, u) \subset B_{C_2 h^q}(x_0), \tag{2-1}$$

where q depends only on n, λ, and Λ, and C_1, C_2, and h_0 depend only on n, λ, Λ, and $dist(\Omega', \partial\Omega)$.

Proof. Let $\varepsilon = dist(\Omega', \partial\Omega)$. By [Gu, Theorem 5.3.3], there exists h_0 such that for any $x_0 \in \Omega'$

$$S_{h_0}(x_0, u) \subset \{x \in \Omega : u(x) \le -C\varepsilon\}.$$

We claim there exists $\frac{1}{2} < \theta = \theta(n, \lambda, \Lambda) < 1$ such that for $x_0 \in \Omega'$ and $0 < h \le h_0$

$$\frac{1}{2} S_h(x_0, u) \subset S_{h/2}(x_0, u) \subset \theta S_h(x_0, u), \tag{2-2}$$

where the dilations are with respect to x_0. Indeed, the first inclusion follows from the convexity of u, and the proof of $S_{h/2}(x_0, u) \subset \theta S_h(x_0, u)$ is similar to that of [Gu, Theorem 3.3.5] with the dilations with respect to x_0 replacing the center of mass.

To finish the proof, it suffices to prove (2-1) with $h = 2^{-i} h_0$ for $i \ge 1$. Applying (2-2) repeatedly yields

$$2^{-i} S_{h_0}(x_0, u) \subset S_{2^{-i}h_0}(x_0, u) \subset \theta^i S_{h_0}(x_0, u), \qquad \text{for } i \ge 1.$$

Since $S_{h_0}(x_0, u) \subset \Omega$, $\theta^i S_{h_0}(x_0, u) \subset B_{2\theta^i}(x_0)$. On the other hand, by the Alexandrov estimate and [Gu, Corollary 3.2.4], $dist(x_0, \partial S_{h_0}(x_0, u)) \ge C h_0^{n/2}$. Thus, $2^{-i} S_{h_0}(x_0, u) \supset B_{C h_0^{n/2} 2^{-i}}(x_0)$. We have proved (2-1) for $h = 2^{-i} h_0$. $\square$

Example 2.2. Let $\Omega_1 = \{(x,y) \in \mathbb{R}^2 : |y| < |x|^3\}$ and $\Omega_2 = \mathbb{R}^2 \setminus \overline{\Omega}_1$. Consider

$$u(x,y) = \begin{cases} x^4 + \dfrac{3}{2}\dfrac{y^2}{x^2} & \text{if } (x,y) \in \overline{\Omega}_1, \\ \dfrac{1}{2}x^2 y^{2/3} + 2y^{4/3} & \text{if } (x,y) \in \Omega_2. \end{cases}$$

It was proved in [W] that $u \in C^{1,\frac{1}{3}}$, $u \notin W^{2,p}$ for $p \geq 2$, and u is a strictly convex weak solution to

$$\det D^2 u = f(x,y), \qquad \text{in } \mathbb{R}^2,$$

where

$$f(x,y) = \begin{cases} 36 - 9y^2 x^{-6} & \text{in } \Omega_1, \\ \dfrac{8}{9} - \dfrac{5}{9}x^2 y^{-2/3} & \text{in } \Omega_2. \end{cases}$$

Obviously, $27 \leq f(x,y) \leq 36$ in Ω_1, and $\dfrac{1}{3} \leq f(x,y) \leq \dfrac{8}{9}$ in Ω_2. In fact, this example from [W] reveals a difference between the Monge-Ampère equation and fully nonlinear uniformly elliptic equations when considering the $W^{2,p}$ theory. Since f has a jump discontinuity at the curves $|y| = |x|^3$, $f \notin \text{VMO}(\mathbb{R}^2)$. Below we further show $f \notin \text{VMO}_{loc}(\mathbb{R}^2, u)$.

Proof. Consider the sections $S_h(0,u) = \{(x,y) : u(x,y) < h\}$. It is easy to see that there exists a constant $0 < \delta < 1$ such that

$$R_{\delta h} \subset S_h(0,u) \subset R_h,$$

where $R_h = \{(x,y) : |x| \leq h^{1/4}, |y| \leq h^{3/4}\}$. Set $R_h^i = R_h \cap \Omega_i$ for $i = 1, 2$. It is easy to verify that $|R_h^1| = h = \dfrac{1}{4}|R_h|$, $|R_h^2| = \dfrac{3}{4}|R_h|$, and

$$7 \leq \fint_{R_h} f\, dxdy \leq \frac{29}{3}.$$

Therefore for $h > 0$, we have $\text{mosc}_{R_h} f \geq 13/3$ and $\text{mosc}_{S_h(0,u)} f \geq C\delta \text{mosc}_{R_{\delta h}} f \geq C\delta$. So $f \notin \text{VMO}_{loc}(\mathbb{R}^2, u)$. $\square$

We now give some discontinuous functions which are in certain $\text{VMO}_{loc}^{\psi}(\Omega, u)$.

Example 2.3. Let $g(x) = \left(\log\log\dfrac{1}{|x|}\right)^{\alpha}$ for $x = (x_1, x_2) \in B_{r_0}$, where r_0 is small and $0 < \alpha < 1$. Let $R \subset B_{r_0}$ be a rectangle of length λ_1 and width λ_2 such that $\lambda_2 \leq \lambda_1 \leq K\lambda_2^{\theta}$ for some $K > 1$ and $0 < \theta < 1$. Then we have

$$\text{mosc}_R^{(2)} g = \left(\fint\!\!\fint_R |g - g_R|^2\right)^{1/2} \leq C\left(\log\log\frac{1}{\lambda_1}\right)^{\alpha-1}. \tag{2-3}$$

Consequently, $\operatorname{mosc}_R^{(2)} g(r_0 x) \le C(\log\log(8/\lambda_1))^{\alpha-1}$ for any rectangle $R \subset B_1$ of length λ_1 and width λ_2 satisfying $\lambda_2 \le \lambda_1 \le K\lambda_2^\theta$. Let u, f, and Ω be as in Lemma 2.1, and $n = 2$. For $\Omega' \subset\subset \Omega$, $x_0 \in \Omega'$, and $h \le h_0$, by Fritz John's lemma, there exists a rectangle R such that $\alpha_2' R \subset S_h(x_0, u) \subset R$ with the dilation with respect to the center of R. By Lemma 2.1, the length λ_1 and width λ_2 of R satisfy $C_1 h \le \lambda_2 \le \lambda_1 \le C_2 h^q$. Therefore

$$\operatorname{mosc}_{S_h(x_0,u)}^{(2)} g(r_0 x) \le C\left(\log\log \frac{8}{diam(S_h(x_0,u))}\right)^{\alpha-1}. \tag{2-4}$$

One concludes $g(r_0 x) \in \mathrm{VMO}_{loc}^{\psi}(\Omega, u)$ where $\psi(t) = (\log\log(8/t))^{\alpha-1}$.

Consider $F(g(r_0 x))$, where $F(t)$ is globally Lipschitz on $[0, \infty)$ and $\lambda \le F(t) \le \Lambda$. Let u be the solution to $\det D^2 u = F(g(r_0 x))$ in Ω with $u = 0$ on $\partial\Omega$, where $B_{\alpha_2} \subset \Omega \subset B_1$. Then by (2-4), $F(g(r_0 x)) \in \mathrm{VMO}_{loc}^{\psi}(\Omega, u)$. In general, $F(g(r_0 x))$ is discontinuous at $x = 0$.

Proof of (2-3). Since g is invariant under rotations and reflections, one may assume that $R = [a - \lambda_1, a + \lambda_1] \times [b - \lambda_2, b + \lambda_2]$, $a, b \ge 0$. To proceed, we distinguish five cases.

Case I: $a = b = 0$. By the transform $(y_1, y_2) = (\lambda_1 x_1, \lambda_2 x_2)$ and the Sobolev inequality

$$\begin{aligned}\operatorname{mosc}_R^{(2)} g &\le C\left(\log\log\frac{1}{\lambda_1}\right)^{\alpha-1}\left(\log\frac{1}{\lambda_1}\right)^{-1}\int_0^1\int_0^1 \frac{\lambda_1^2 x_1 + \lambda_2^2 x_2}{\lambda_1^2 x_1^2 + \lambda_2^2 x_2^2}\,dx_1 dx_2\\ &\le C\left(\log\log\frac{1}{\lambda_1}\right)^{\alpha-1}\left(\log\frac{1}{\lambda_1}\right)^{-1}\left(C + \log\frac{\lambda_1}{\lambda_2}\right).\end{aligned}$$

Then (2-3) follows.

Case II: $a = 0$, $b < 2\lambda_2$. This case is reduced to case I, because $[b - \lambda_2, b + \lambda_2] \subset [-3\lambda_2, 3\lambda_2]$.

Case III: $a = 0$, $b \ge 2\lambda_2$. Note $[b - \lambda_2, b + \lambda_2] \subset [b/2, 3b/2]$. Similar to case I, we have

$$\begin{aligned}\operatorname{mosc}_R^{(2)} g &\le C\left(\log\log\frac{1}{\lambda_1 + b}\right)^{\alpha-1}\left(\log\frac{1}{\lambda_1 + b}\right)^{-1}\int_0^1\int_0^1 \frac{\lambda_1^2 x_1 + \lambda_2 b}{\lambda_1^2 x_1^2 + b^2}\,dx_1 dx_2\\ &\le C\left(\log\log\frac{1}{\lambda_1 + b}\right)^{\alpha-1}\left(\log\frac{1}{\lambda_1 + b}\right)^{-1}\left(\frac{\lambda_2}{b} + \log(1 + \lambda_1^2/b^2)\right).\end{aligned}$$

Then (2-3) can be obtained by considering the cases $2\lambda_2 \le b < \sqrt{\lambda_1}$ and $b \ge \sqrt{\lambda_1}$.

Case IV: $a \le 2\lambda_1$. This case can be reduced to cases I-III, since $[a - \lambda_1, a + \lambda_1] \subset [-3\lambda_1, 3\lambda_1]$.

Case V: $a > 2\lambda_1$. By the mean value theorem

$$\operatorname{osc}_R g \le C \sup_R \left|\left(\log\log\frac{1}{|x|}\right)^{\alpha-1}\left(\log\frac{1}{|x|}\right)^{-1}\frac{1}{|x|}\right| \cdot \lambda_1.$$

By distinguishing two cases i) $a \ge \sqrt{\lambda_1}$ or $b \ge \sqrt{\lambda_1}$ and ii) $a < \sqrt{\lambda_1}$ and $b < \sqrt{\lambda_1}$, we obtain $\operatorname{osc}_R g \le C(\log\log\frac{1}{\lambda_1})^{\alpha-1}(\log\frac{1}{\lambda_1})^{-1}$. Thus, the proof of (2-3) is complete. □

Remark 2.4. Let $G(x) \in L^n(B_1)$ such that $\operatorname{mosc}_R^{(n)} G \le C\psi(diam(R))$ for any rectangle $R \subset B_1$ whose smallest edgelength λ_m and largest edgelength λ_M satisfy $\lambda_m \le \lambda_M \le K\lambda_m^\theta$, where C depends n, K, and θ. Let u be the solution to $\det D^2 u = f(x)$ in Ω satisfying $u = 0$ on $\partial\Omega$, where $B_{\alpha_n} \subset \Omega \subset B_1$ and $\lambda \le f \le \Lambda$ in Ω. Then $G \in \mathrm{VMO}_{loc}^{\psi}(\Omega, u)$.

§3. A comparison principle

Consider the operator $\mathcal{M}u = (\det D^2u)^{1/n}$ and its linearized operator

$$L_u v = \frac{1}{n}(\det D^2 u)^{1/n} tr\left((D^2u)^{-1}D^2v\right). \tag{3-1}$$

We have the following maximum principle for the operator L_u: if u is a strongly convex C^2 function and $v \in C^2(\overline{\Omega})$ then

$$\max_{\overline{\Omega}} v \le \max_{\partial\Omega} v + c_n diam(\Omega)\left(\int_{\Gamma_v}(-L_u v)^{+n}\,dx\right)^{1/n}, \tag{3-2}$$

where $\Gamma_v = \{x \in \Omega : D^2v \le 0\}$ and c_n is a dimensional constant.

To prove (3-2), recall the Alexandrov-Bakelman-Pucci (A-B-P) maximum principle

$$\max_{\overline{\Omega}} v \le \max_{\partial\Omega} v + c_n diam(\Omega)\left(\int_{\Gamma_v}|\det D^2v|\,dx\right)^{1/n}.$$

By the arithmetic-geometric inequality, one obtains on Γ_v

$$\det(-D^2v) \le \left(\frac{1}{n}tr\left((D^2u)^{-1}(-D^2v)\right)\right)^n \det D^2u = (-L_u v)^{+n},$$

which, together with the A-B-P maximum principle, yields (3-2).

Lemma 3.1. *Let u and w be the weak solutions to $\det D^2u = g_1^n \ge 0$ and $\det D^2w = g_2^n \ge 0$ in Ω, respectively. Assume that $g_1,\, g_2 \in L^n(\Omega)$. Then*

$$\max_{\overline{\Omega}}(u-w) \le \max_{\partial\Omega}(u-w) + c_n diam(\Omega)\left(\int_{\Omega}(g_2-g_1)^{+n}\,dx\right)^{1/n}.$$

Proof. Through approximations by smooth solutions, one may assume that u and w are strongly convex C^2 solutions. By the concavity of the operator $\mathcal{M}$, we have $g_1 - g_2 = \mathcal{M}u - \mathcal{M}w \le L_w(u-w)$ in Ω. Lemma 3.1 follows by applying (3-2) to $u-w$. □

A similar result to Lemma 3.1 can be found in [Ji-W]. Lemma 3.1 allows us to compare the solutions u and w even if $g_2 - g_1$ changes the sign. As a corollary, we have

Lemma 3.2. *Let u satisfy $\det D^2u = g^n \ge 0$ in Ω and w satisfy $\det D^2w = 1$ in Ω. Assume $g \in L^n(\Omega)$. Then*

$$\max_{\overline{\Omega}}|u - g_\Omega w| \le \max_{\partial\Omega}|u - g_\Omega w| + c_n diam(\Omega)\left(\int_{\Omega}|g-g_\Omega|^n\,dx\right)^{1/n}.$$

Proof. Apply Lemma 3.1 to both $u - g_\Omega w$ and $g_\Omega w - u$. □

§4. Eccentricity of cross sections

Our goal here is to investigate the eccentricity of sections of solutions to (1-1) if f is in VMO type spaces. These estimates of the eccentricity are important in deriving estimates of D^2u in VMO^ψ spaces and can also give Caffarelli's $C^{2,\alpha}$ estimates and gradient estimates. Since f is just in VMO type spaces, the perturbation behavior of (1-1) is more complicated, and Lemma 3.2 is needed to replace the classical comparison principle. These results are a refinement of those in [H].

In the rest of this paper, we use the symbol $a \approx b$ to denote that the quantity a/b is bounded by two positive universal constants from above and below. $a \ll b$ denotes a/b is bounded by a universal constant much smaller than 1. We use C to denote universal constants dependent only on structure constants. Let $aE = \{ax : x \in E\}$ be the a-dilation of E with respect to the origin unless otherwise stated.

Lemma 4.1. *Let u be a weak solution to (1-1) in Ω and $u = 0$ on $\partial\Omega$. Assume that $B_{a_1} \subset \Omega \subset B_{a_2}$, $\lambda \le f = g^n \le \Lambda$, and $mosc^{(n)}_\Omega g = \left(⨍_\Omega |g - g_\Omega|^n \right)^{1/n} \le \varepsilon$. Then there exist $z_0 \in \Omega$ and a linear transformation A with $\det A = 1$ such that*

$$C^{-1} \le \|Ax\| \le C \qquad \text{for } \|x\| = 1, \tag{4-1}$$

and for small $\mu > 0$ with $\varepsilon \ll \mu$

$$(1-\delta)B_{\sqrt{2/g_\Omega}} \subset \sqrt{\mu}^{\,-1} T S_\mu(u) \subset (1+\delta)B_{\sqrt{2/g_\Omega}}, \tag{4-2}$$

where $\delta = C(\sqrt{\mu} + \varepsilon/\mu)$, $Tx = A(x - z_0)$, and $S_\mu(u) = \{x \in \Omega : u(x) < \min_\Omega u + \mu\}$.

Proof. Let $w(x)$ be the smooth convex solution to $\det D^2 w = 1$ in Ω with $w = 0$ on $\partial\Omega$. By Lemma 3.2 we get

$$|u - g_\Omega w| \le C\varepsilon \qquad \text{in } \Omega. \tag{4-3}$$

By the classical regularity theory of the Monge-Ampère equation, one has interior C^∞ estimates for w.

Let $w(z_0) = \min_\Omega w$, $S_\mu(w) = S_\mu(w, z_0)$, and $u^* = u/g_\Omega$. Recall $|\min_\Omega u| \approx$ *const.* From (4-3), $|\min_\Omega u^* - \min_\Omega w| \le C\varepsilon$ and

$$S_{\mu - C\varepsilon}(w) \subset S_\mu(u^*) \subset S_{\mu + C\varepsilon}(w).$$

As in the proof of [H, Lemma 2.1], we have for small μ

$$\partial S_\mu(u^*) \subset z_0 + N_{C(\mu + \varepsilon/\sqrt{\mu})}\big(\partial \sqrt{\mu} E\big),$$

where $E = \{x : \frac{1}{2} D_{ij} w(z_0) x_i x_j < 1\}$, N_η is the η-neighborhood with respect to the Euclidean distance, and $z_0 + B = \{z_0 + x : x \in B\}$. Since $S_\mu(u) = S_{\mu/g_\Omega}(u^*)$

$$\partial S_\mu(u) \subset z_0 + N_{C(\mu + \varepsilon/\sqrt{\mu})}\big(\partial \sqrt{\mu/g_\Omega} E\big). \tag{4-4}$$

To finish the proof, we write $D^2 w(z_0) = A^2$ for some $A > 0$ because $D^2 w$ is positively definite, and let $Tx = A(x - z_0)$. It is easy to verify that $\det A = 1$ and (4-1) holds. (4-2) follows from (4-4). $\square$

If the shape of Ω is close to that of a ball, then one can refine Lemma 4.1.

Lemma 4.2. *Let u be a weak solution to (1-1) in Ω and $u = 0$ on $\partial\Omega$. Assume that $\lambda \le f = g^n \le \Lambda$, $mosc^{(n)}_{\Omega} g \le \varepsilon$, $(1-\delta)B_{\sqrt{2/b}} \subset \Omega \subset (1+\delta)B_{\sqrt{2/b}}$, and $\lambda \le b^n \le \Lambda$, where ε and δ are small. Then there exist $z_0 \in \Omega$ and a linear transformation A with $\det A = 1$ such that*

$$1 - C\delta \le \|Ax\| \le 1 + C\delta \qquad \text{for } \|x\| = 1, \tag{4-5}$$

and for small $\mu > 0$ with $\varepsilon \ll \mu$

$$(1-\eta)B_{\sqrt{2/g_\Omega}} \subset \sqrt{\mu}^{\,-1} T S_\mu(u) \subset (1+\eta)B_{\sqrt{2/g_\Omega}}, \tag{4-6}$$

where $\eta = C(\delta\sqrt{\mu} + \varepsilon/\mu)$ and $Tx = A(x - z_0)$.

Proof. Let w be the convex smooth solution to

$$\det D^2 w = 1 \qquad \text{in } \Omega, \tag{4-7}$$

and $w = 0$ on $\partial\Omega$. By Lemma 3.2, $\max_{\overline{\Omega}} |u - g_\Omega w| \le C\varepsilon$. Let $P(x) = \frac{1}{2}|x|^2 - \frac{1}{b}$. Clearly P satisfies the equation (4-7). Since $\partial\Omega \subset N_{C\delta}(\partial B_{\sqrt{2/b}})$, $|P| \le C\delta$ on $\partial\Omega$. By the comparison principle

$$\max_{\overline{\Omega}} |w - P| \le C\delta.$$

Observe that $v = w - P$ satisfies the uniformly elliptic linear equation $\mathrm{tr}(D(x)D^2 v) = 0$, where $D(x) = \int_0^1 \det(\theta D^2 w + (1-\theta)I)(\theta D^2 w + (1-\theta)I)^{-1}\, d\theta$ and I is the $n \times n$ unit matrix, and that by differentiating (4-7), $D_k(w - P)$ satisfies the linearized equation $\mathrm{tr}((D^2 w)^{-1} D^2[D_k(w-P)]) = 0$. Applying the interior Schauder estimates to $w - P$ and $D_k(w-P)$, we obtain

$$\|w - P\|_{C^3_{\mathrm{loc}}} \le C\|w - P\|_{L^\infty} \le C\delta.$$

Therefore, $\|D^3 w\|_{L^\infty_{\mathrm{loc}}} \le C\delta$ and $|D^2 w(z_0) - I| \le C\delta$, where $w(z_0) = \min_\Omega w$.

Similar to the proof of [H, Lemma 2.2], we have

$$\partial S_\mu(u^*) \subset z_0 + N_{C(\delta\mu + \varepsilon/\sqrt{\mu})}\left(\partial\sqrt{\mu}E\right).$$

where $u^* = u/g_\Omega$ and $E = \{x : \frac{1}{2}D_{ij}w(z_0)x_i x_j < 1\}$. It follows that

$$\partial S_\mu(u) \subset z_0 + N_{C(\delta\mu + \varepsilon/\sqrt{\mu})}\left(\partial\sqrt{\mu/g_\Omega}E\right). \tag{4-8}$$

Let $D^2 w(z_0) = A^2$ with $A > 0$. Then $\det A = 1$ and (4-5) follows from $|D^2 w(z_0) - I| \le C\delta$. It is easy to see that (4-8) yields (4-6). $\square$

Lemma 4.3. *Suppose that the assumptions in Lemma 4.1 hold and furthermore there is a sequence* $\{\varepsilon_k\}_{k=0}^{\infty}$ *with* $0 < \varepsilon_{k+1} \le \varepsilon_k$ *and* $\varepsilon_0 = \varepsilon \ll \mu$ *such that* $\mathrm{mosc}^{(n)}_{S_{\mu^k}(u)} g \le \varepsilon_k$ *for* $k \ge 1$*. Then there exist* $\{z_k\} \subset \mathbb{R}^n$ *and linear transformations* $\{A_k\}$ *such that*

$$\det A_k = 1, \quad \textit{for } k \ge 1$$
$$C^{-1} \le \|A_1 x\| \le C, \quad \textit{for } \|x\| = 1$$
$$1 - C\delta_{k-1} \le \|A_k x\| \le 1 + C\delta_{k-1}, \quad \textit{for } k \ge 2,\ \|x\| = 1$$
$$(1-\delta_k) B_{\sqrt{2/b_k}} \subset \mu^{-1/2} T_k \cdots \mu^{-1/2} T_1 S_{\mu^k}(u) \subset (1+\delta_k) B_{\sqrt{2/b_k}},$$

where $T_k x = A_k(x - z_k)$*,* $\delta_0 = 1$ *and* $\delta_k = C(\delta_{k-1}\sqrt{\mu} + \varepsilon_{k-1}/\mu)$ *for* $k \ge 1$*,* $b_1 = g_\Omega$ *and* $b_k = g_{S_{\mu^{k-1}}(u)}$ *for* $k \ge 2$*.*

Moreover, if $u(x_0) = \min_\Omega u$*, then*

$$B_{C^{-1}} \subset \mu^{-k/2} A_k \cdots A_1 [S_{\mu^k}(x_0, u) - x_0] \subset B_C. \tag{4-9}$$

Proof. By Lemma 4.1, there exist z_1 and A_1 with $\det A_1 = 1$ such that

$$C^{-1} \le \|A_1 x\| \le C \quad \text{for } \|x\| = 1,$$
$$(1-\delta_1) B_{\sqrt{2/b_1}} \subset \mu^{-1/2} T_1 S_\mu(u) \subset (1+\delta_1) B_{\sqrt{2/b_1}}.$$

Let $u_1(x) = \mu^{-1}\left[u(T_1^{-1}\sqrt{\mu}x) - (\min_\Omega u + \mu)\right]$ and $\Omega_1 = \sqrt{\mu}^{-1} T_1 S_\mu(u)$. It is easy to verify that

$$\det D^2 u_1 = g_1^n(x) = g^n(T_1^{-1}\sqrt{\mu}x) \qquad \text{in } \Omega_1,$$

$u_1 = 0$ on $\partial\Omega_1$, and $\mathrm{mosc}^{(n)}_{\Omega_1} g_1 = \mathrm{mosc}^{(n)}_{S_\mu(u)} g \le \varepsilon_1$. By applying Lemma 4.2 to u_1 in Ω_1, there exist z_2 and a linear transformation A_2 with $\det A_2 = 1$ such that

$$1 - C\delta_1 \le \|A_2 x\| \le 1 + C\delta_1, \quad \text{for } \|x\| = 1$$
$$(1-\delta_2) B_{\sqrt{2/b_2}} \subset \sqrt{\mu}^{-1} T_2 S_\mu(u_1, \Omega_1) = \mu^{-1/2} T_2 \mu^{-1/2} T_1 S_{\mu^2}(u) \subset (1+\delta_2) B_{\sqrt{2/b_2}},$$

where $b_2 = (g_1)_{\Omega_1} = g_{S_\mu(u)}$ and $S_\mu(u_1, \Omega_1) = \{x \in \Omega_1 : u_1(x) < \min_{\Omega_1} u_1 + \mu\}$. Proceed by induction. Assume that the conclusions in the lemma (except (4-9)) are valid for the cases up to $k \ge 2$. Consider the normalized solution and domain given by

$$u_k(x) = \mu^{-k}\left[u(T_1^{-1}\sqrt{\mu} \cdots T_k^{-1}\sqrt{\mu}x) - (\min_\Omega u + \mu^k)\right]$$

and $\Omega_k = \mu^{-1/2} T_k \cdots \mu^{-1/2} T_1 S_{\mu^k}(u)$. One can easily check that u_k satisfies

$$\det D^2 u_k = g_k^n(x) = g^n(T_1^{-1}\sqrt{\mu} \cdots T_k^{-1}\sqrt{\mu}x) \qquad \text{in } \Omega_k,$$

$u_k = 0$ on $\partial\Omega_k$, and $\operatorname{mosc}^{(n)}_{\Omega_k} g_k = \operatorname{mosc}^{(n)}_{S_{\mu^k}(u)} g \le \varepsilon_k$. By the induction hypotheses and applying Lemma 4.2 to u_k in Ω_k, there exist z_{k+1} and a linear transfromation A_{k+1} with $\det A_{k+1} = 1$ such that

$$\begin{aligned} 1 - C\delta_k \le \|A_{k+1}x\| &\le 1 + C\delta_k, \quad \text{for } \|x\| = 1 \\ (1-\delta_{k+1})B_{\sqrt{2/b_{k+1}}} &\subset \sqrt{\mu}^{\,-1} T_{k+1} S_\mu(u_k, \Omega_k) \\ &= \mu^{-1/2} T_{k+1} \cdots \mu^{-1/2} T_1 S_{\mu^{k+1}}(u) \subset (1+\delta_{k+1}) B_{\sqrt{2/b_{k+1}}}, \end{aligned}$$

where $b_{k+1} = (g_k)_{\Omega_k} = g_{S_{\mu^k(u)}}$.

To prove (4-9), note $\mathcal{T}_k x = \mu^{-1/2} T_k \cdots \mu^{-1/2} T_1 x = \mu^{-k/2} A_k \cdots A_1 x + y_k$ for some y_k. If we set $y_0 = \mathcal{T}_k x_0$, then $\mathcal{T}_k x = \mu^{-k/2} A_k \cdots A_1(x - x_0) + y_0$. Obviously, $u_k(y_0) = \min_{\Omega_k} u_k = -1$. Applying the Alexandrov estimate to u_k in Ω_k, one concludes that y_0 must lie strictly inside Ω_k. Therefore, there exists $C > 0$ such that

$$B_{C^{-1}}(y_0) \subset \Omega_k \subset (1+\delta_k) B_{\sqrt{2/b_k}} \subset B_C(y_0).$$

It yields (4-9), and Lemma 4.3 is proved. □

§5 Regularity from the eccentricity

In this section, we first use the estimates of eccentricity in §4 to give $C^{1,\alpha}$ estimates if $f \in \mathrm{VMO}_{loc}(\Omega, u)$. $C^{1,\alpha}$ estimates for any $0 < \alpha < 1$ were first obtained by Caffarelli for continuous f. Second, we give Caffarelli's $C^{2,\alpha}$ estimates by directly studying sections. Alternatively, the $C^{2,\alpha}$ estimates are the special case of Theorem B(ii) with $\psi(t) = t^\alpha$.

Theorem 5.1. *Let u be a convex weak solution to* (1-1) *in Ω satisfying $u = 0$ on $\partial\Omega$, where $B_{\alpha_n} \subset \Omega \subset B_1$ is a convex domain. Assume that $\lambda \le f(x) \le \Lambda$ in Ω and $f \in VMO_{loc}(\Omega, u)$. Then $u \in C^{1,\alpha}_{loc}(\Omega)$ for any $0 < \alpha < 1$.*

Proof. By Lemma 2.1 and since $\lambda \le f = g^n \le \Lambda$, given $\Omega' \subset\subset \Omega$ and $\varepsilon_0 > 0$, there exists h_0 such that for $x_0 \in \Omega'$, $0 < h \le h_0$

$$\operatorname{mosc}^{(n)}_{S_h(x_0,u)} g \le C \operatorname{mosc}^{(n)}_{S_h(x_0,u)} f \le C[Q_f(Ch^q, \Omega')]^{1/n} \le \varepsilon_0. \tag{5-1}$$

We now normalize u and $S_{h_0}(x_0, u)$. Let E be the Fritz John ellipsoid of $S_{h_0}(x_0, u)$ and T be the affine transformation such that $Tx = A(x - x_0) + y_0$ and $TE = B_1$. Set

$$u^*(y) = |\det A|^{2/n} \left((u - \ell_{x_0})(T^{-1}y) - h_0\right), \tag{5-2}$$

where $\ell_{x_0}(x)$ is the supporting function of u at x_0. It is easy to verify that u^* satisfies

$$\det D^2 u^* = f^*(y) = (g^*(y))^n \qquad \text{in } S^* = TS_{h_0}(x_0, u), \tag{5-3}$$

where $f^*(y) = f(T^{-1}y)$ and $B_{1/n} \subset S^* \subset B_1$. Obviously, $S_h(y_0, u^*) = TS_{C_A h}(x_0, u)$ for $h \le h_0/C_A$, where $C_A = |\det A|^{-2/n}$ and $h_0/C_A = |\min_{S^*} u^*| \approx const$. From (5-1), we have for $h \le h_0/C_A$

$$\operatorname{mosc}^{(n)}_{S_h(y_0,u^*)} g^* = \operatorname{mosc}^{(n)}_{S_{C_A h}(x_0,u)} g \le \varepsilon_0.$$

By applying Lemma 4.3 with all $\varepsilon_k = \varepsilon_0$ to u^*, there exists $\{A_k\}$ such that

$$B_{C^{-1}} \subset \mu^{-k/2} A_k \cdots A_1 [S_{\mu^k}(y_0, u^*) - y_0] \subset B_C.$$

Choose μ small and $\varepsilon_0 \le \mu^2$. Then $\delta_1 \le C\sqrt{\mu}$ and $\delta_{k+1} < \delta_k$ by induction. So $\|A_k\| \le 1+\sigma$ and $\|A_k^{-1}\| \le 1 + \sigma$ for $k \ge 2$, where $\sigma = C\sqrt{\mu}$. Therefore

$$B_{C^{-1}(1-\sigma)^k \sqrt{\mu}^k}(y_0) \subset S_{\mu^k}(y_0, u^*) \subset B_{C(1+\sigma)^k \sqrt{\mu}^k}(y_0). \tag{5-4}$$

To show $C^{1,\alpha}$ estimates, let $y \in S_{\mu^k}(y_0, u^*) \setminus S_{\mu^{k+1}}(y_0, u^*)$, i.e., $\mu^{k+1} \le u^*(y) - u^*(y_0) \le \mu^k$. From (5-4)

$$C^{-1}\left((1-\sigma)\sqrt{\mu}\right)^{k+1} \le |y - y_0| \le C\left((1+\sigma)\sqrt{\mu}\right)^k.$$

Write $1 - \sigma = \mu^\tau$. Then $\mu^{(\frac{1}{2}+\tau)(k+1)} \le C|y - y_0|$. Hence

$$|u^*(y) - u^*(y_0)| \le C\mu^{-1}|y - y_0|^{\frac{2}{1+2\tau}} = C_\mu |y - y_0|^{1+\alpha},$$

where $\alpha \to 1$ as $\mu \to 0$. Since $u^*(y_0) = \min_{S^*} u^*$, u^* is $C^{1,\alpha}$ at y_0. Therefore, $|u(x) - \ell_{x_0}(x)| \le C_\mu C_A \|A\|^{1+\alpha}|x - x_0|^{1+\alpha}$ on $S_{h_0}(x_0, u)$ and u is $C^{1,\alpha}$ at x_0. □

Lemma 5.2. *Let u be a weak solution to (1-1) in Ω and $u = 0$ on $\partial\Omega$. Assume that $B_{a_1} \subset \Omega \subset B_{a_2}$ and $\lambda \le f = g^n \le \Lambda$. Let $u(x_0) = \min_\Omega u$ and μ is small. Suppose that $mosc^{(n)}_\Omega g \le \omega(2a_2) \ll \mu$ and $mosc^{(n)}_S g \le \omega(diam(S))$ for $S = S_{\mu^k}(x_0, u)$, $k \ge 1$, where $\omega(0) = 0$, ω is increasing and $r^\gamma/\omega(r)$ is almost increasing in $(0, 2a_2]$ for some $0 < \gamma < 1$. Then there exist δ_k decreasing to 0 with $\delta_1 \ll 1$ and linear transforms A_k with* $\det A_k = 1$ *such that*

$$C^{-1} \le \|A_1 x\| \le C \quad \text{for } \|x\| = 1,$$
$$1 - C\delta_{k-1} \le \|A_k x\| \le 1 + C\delta_{k-1}, \quad \text{for } k \ge 2,\ \|x\| = 1,$$

and for $k \ge 1$, $\mathcal{T}_k = A_k \cdots A_1$ satisfies

$$C^{-1}e^{-C_\mu \Phi(\sqrt{\mu}^k)} \le \|\mathcal{T}_k x\| \le Ce^{C_\mu \Phi(\sqrt{\mu}^k)}, \quad \text{for } \|x\| = 1,$$
$$B_{C^{-1}} \subset \mu^{-k/2}\mathcal{T}_k[S_{\mu^k}(x_0, u) - x_0] \subset B_C,$$

where $\Phi(r) = \int_r^{2a_2} \frac{\omega(t)}{t}\,dt$ and C_μ is a constant dependent on μ.

The proof of Lemma 5.2 is the same as that of [H, Lemma 2.5] by using Lemma 4.3 and estimating δ_k, and we skip it.

Lemma 5.2 is one of ingredients in the proof of Theorem B and also yields other regularity results. Generally, it can give gradient estimates better than any $C^{1,\alpha}$ close to those in [Ji-W, Theorem 2] for continuous $f(x)$. Particularly, it gives $C^{1,1}$ estimate if ω satisfies a Dini condition. We below show that if $f \in C^\alpha$, then cross sections have even better shape and one then obtains $C^{2,\alpha}$ estimates.

Lemma 5.3. *Let u be a weak solution to (1-1) in Ω and $u = 0$ on $\partial\Omega$. Assume that $B_{a_1} \subset \Omega \subset B_{a_2}$. Let $u(x_0) = \min_\Omega u$ and μ is small. Suppose that $[f]_{C^\alpha(\overline{\Omega})} \le \varepsilon \ll \mu^{3/2}$ for some $0 < \alpha < 1$ and $f(x_0) = 1$. Then there exist a linear transform A satisfying* $\det A = 1$ *and $\|A\| \le C$ and such that for $k \ge 1$*

$$(1 - C\mu^{k\alpha/4})B_{\sqrt{2}} \subset \mu^{-k/2}A[S_{\mu^k}(x_0, u) - x_0] \subset (1 + C\mu^{k\alpha/4})B_{\sqrt{2}}. \tag{5-5}$$

Proof. By [H, Lemma 2.3], there exist linear transforms A_k with $\det A_k = 1$ such that

$$(1 - \delta_k)B_{\sqrt{2}} \subset \mathcal{B}_k S_{\mu^k}(x_0, u) \subset (1 + \delta_k)B_{\sqrt{2}}, \tag{5-6}$$

where $\mathcal{B}_k x = \mu^{-k/2}A_k \cdots A_1(x - x_0) + y_k$ for some y_k, $\delta_k = C(\delta_{k-1}\sqrt{\mu} + \varepsilon_{k-1}/\mu)$, and $\varepsilon_k = osc_{S_{\mu^k}(x_0,u)} f$. We have $\|\mathcal{T}_k^{-1}\| \le C$ and $diam(S_{\mu^k}(x_0, u)) \le C\sqrt{\mu}^k$ according to [H, Lemma 2.5], where $\mathcal{T}_k = A_k \cdots A_1$. Therefore $\varepsilon_k \le C\varepsilon\sqrt{\mu}^{k\alpha}$, and we have for $k \ge 1$

$$\delta_k = \frac{C}{\mu}\sum_{i=0}^{k-1}(C\sqrt{\mu})^{k-1-i}\varepsilon_i + (C\sqrt{\mu})^k \le \sqrt{\mu}^{k\alpha}.$$

Let $u_k^* = \mu^{-k}[u(\mathcal{B}_k^{-1}y) - u(x_0) - \mu^k]$. Obviously

$$1 - \varepsilon_k \le \det D^2 u_k^* = f(\mathcal{B}_k^{-1}y) \le 1 + \varepsilon_k \qquad \text{in } S_k^* = \mathcal{B}_k S_{\mu^k}(x_0, u),$$

and $u_k^* = 0$ on ∂S_k^*. By (5-6) and the comparison principle

$$(1 + \varepsilon_k)(\tfrac{1}{2}|y|^2 - 1 - 3\delta_k) \le u_k^*(y) \le (1 - \varepsilon_k)(\tfrac{1}{2}|y|^2 - 1 + 3\delta_k) \qquad \text{in } S_k^*.$$

Noting $u_k^*(y_k) = -1$, we obtain $|y_k| \le C\sqrt{\delta_k + \varepsilon_k} \le C\mu^{k\alpha/4}$. Therefore from (5-6)

$$(1 - C\mu^{k\alpha/4})B_{\sqrt{2}} \subset \mu^{-k/2}\mathcal{T}_k[S_{\mu^k}(x_0, u) - x_0] \subset (1 + C\mu^{k\alpha/4})B_{\sqrt{2}}. \tag{5-7}$$

Notice that $\|A_{k+1} - I\| \le C\delta_k$ by the proofs in [H]. So $\|\mathcal{T}_{k+1} - \mathcal{T}_k\| \le C\delta_k$ and $A = \lim_{k\to\infty}\mathcal{T}_k$ is a well defined linear transform. One can verify that $\det A = 1$, $\|A\| \le C$, $\|A^{-1}\| \le C$, and $\|\mathcal{T}_k - A\| \le C\mu^{k\alpha/2}$ for $k \ge 1$. Since $\mathcal{T}_k A^{-1} = I + (\mathcal{T}_k - A)A^{-1}$, for large M, $\mathcal{T}_k A^{-1}(1 - M\mu^{k\alpha/4})B_{\sqrt{2}} \subset (1 - C\mu^{k\alpha/4})B_{\sqrt{2}}$. Thus, (5-5) follows from (5-7). □

We now use Lemma 5.3 to prove Caffarelli's $C^{2,\alpha}$ estimates.

Theorem 5.4. *Let u be a convex weak solution to* (1-1) *in Ω satisfying $u = 0$ on $\partial\Omega$, where $B_{\alpha_n} \subset \Omega \subset B_1$ is a convex domain. Assume that $\lambda \le f(x) \le \Lambda$ in Ω and $f \in C^\alpha(\overline{\Omega})$. Then $u \in C^{2,\alpha}_{loc}(\Omega)$.*

Proof. Given $x_0 \in \Omega' \subset\subset \Omega$, we first show that u is $C^{2,\alpha/2}$ at x_0. Without loss of generality, one can assume that $u(x_0) = \min_\Omega u$, $f(x_0) = 1$, and $[f]_{C^\alpha(\overline{\Omega})} \le \varepsilon \ll \mu^{3/2}$ for small μ.

Otherwise, as in the proof of Theorem 5.1, we can replace u and Ω by the normalizations of u and $S_{h_0}(x_0,u)$ for some small h_0.

We can apply Lemma 5.3 to u. Let $u^*(y)=u(A^{-1}y)$ and $y_0=Ax_0$, where A is the transform in Lemma 5.3. Clearly, $AS_{\mu^k}(x_0,u)=S_{\mu^k}(y_0,u^*)$. We have $|f(A^{-1}y)-1|\le \varepsilon_k=C\varepsilon\mu^{k\alpha/2}$ for $y\in S_{\mu^k}(y_0,u^*)$. By (5-5) and the comparison principle

$$\begin{aligned}(1+\varepsilon_k)[\tfrac12|y-y_0|^2-\mu^k(1+C\mu^{k\alpha/4})]&\le u^*(y)-u^*(y_0)-\mu^k\\&\le(1-\varepsilon_k)[\tfrac12|y-y_0|^2-\mu^k(1-C\mu^{k\alpha/4})]\end{aligned}$$

for $y\in S_{\mu^k}(y_0,u^*)$. Simple calculations yield

$$\left|u^*(y)-u^*(y_0)-\tfrac12|y-y_0|^2\right|\le C\mu^{k(1+\alpha/4)}\qquad\text{in } S_{\mu^k}(y_0,u^*).$$

Notice that from (5-5), $|y-y_0|\ge C\sqrt{\mu}^{k+1}$ if $y\notin S_{\mu^{k+1}}(y_0,u^*)$. We have

$$\left|u^*(y)-u^*(y_0)-\tfrac12|y-y_0|^2\right|\le C(|y-y_0|/\sqrt{\mu})^{2+\alpha/2}\qquad\text{in } A\Omega.$$

So u is $C^{2,\alpha/2}$ at x_0. Therefore $u\in C^{2,\alpha/2}_{loc}(\Omega)$.

Once $u\in C^{2,\alpha/2}_{loc}(\Omega)$, we can improve (5-5). To do so, assume $\|u\|_{C^{2,\alpha/2}(\overline{\Omega})}\le C$ in Lemma 5.3. As in the proof of Lemma 5.3, if $P(y)=\frac12|y|^2-1$, then $v=u_k^*-P$ satisfies the linearized Monge-Ampère equation

$$\operatorname{tr}(D(x)D^2v)=f(\mathcal{B}_k^{-1}y)-1\qquad\text{in } S_k^*,$$

where $D(x)=\int_0^1\det(\theta D^2u_k^*+(1-\theta)I)(\theta D^2u_k^*+(1-\theta)I)^{-1}\,d\theta$. It is easy to verify that $\|u_k^*-P\|_{L^\infty(S_k^*)}\le C\mu^{k\alpha/2}$, $\|f(\mathcal{B}_k^{-1}y)-1\|_{C^\alpha(\overline{S_k^*})}\le C\varepsilon\mu^{k\alpha/2}$, and $[D^2u_k^*]_{C^{\alpha/2}(\overline{S_k^*})}\le C\mu^{k\alpha/4}\le C$. By the interior Schauder estimates, $\|u_k^*-P\|_{C^{2,\alpha/2}_{loc}}\le C\mu^{k\alpha/2}$. Because $D(u_k^*-P)(y_k)=-y_k$, $|y_k|\le C\mu^{k\alpha/2}$. Following the proof of Lemma 5.3, we obtain the following refinement of (5-5)

$$(1-C\mu^{k\alpha/2})B_{\sqrt2}\subset\mu^{-k/2}A[S_{\mu^k}(x_0,u)-x_0]\subset(1+C\mu^{k\alpha/2})B_{\sqrt2}. \tag{5-8}$$

Repeating the above argument for $C^{2,\alpha/2}_{loc}$ estimates and using (5-8) in place of (5-5), we can show $u\in C^{2,\alpha}_{loc}(\Omega)$. $\square$

§6. $W^{2,p}$ and Orlicz estimates

Our goal in this section is to establish the L^p and Orlicz estimates in Theorem A for the second derivatives of solutions to (1-1). $W^{2,p}$ estimates were obtained by Caffarelli for continuous f in [C3]. Our way to implement the Calderón-Zygmund decomposition is similar to that presented in [Gu]. Since f is in VMO type spaces, there is a difficulty to use convex envelopes of functions to help estimate the density of good sets where the Hessian is bounded. We instead use the idea in [H-L] to deal with it. The Orlicz estimates are derived through a more careful study of the distribution function of D^2u.

We first give the estimate of the density of good sets for the normalized problem.

Lemma 6.1. *Let u be a weak solution to (1-1) in Ω and $u = 0$ on $\partial\Omega$, where $B_{\alpha_n} \subset \Omega \subset B_1$ is a convex domain. Suppose that $\lambda \le f = g^n \le \Lambda$ and $mosc_{\Omega}^{(n)} f \le \varepsilon$. For $0 < \alpha < 1$, let $\Omega_\alpha = \{x \in \Omega : u(x) < (1-\alpha)\min_\Omega u\}$. Then there exist $0 < \delta_0 = \delta_0(\alpha) < 1$ and $\sigma = \sigma(\alpha) > 0$ such that*

$$|\Omega_\alpha \backslash A_\sigma| \le \varepsilon^{\delta_0}|\Omega_\alpha|,$$

where

$$A_\sigma(u) = A_\sigma = \{x_0 \in \Omega : u = P_\sigma \text{ at } x_0 \text{ and } u \ge P_\sigma \text{ in } \Omega\}$$

and $P_\sigma(x) = \sigma|x - x_0|^2 + \ell(x)$ for some affine function $\ell(x)$.

Proof. Let w be the smooth solution to

$$\det D^2 w = 1 \quad \text{in } \Omega, \tag{6-1}$$

and satisfy $w = 0$ on $\partial\Omega$. By Lemma 3.2

$$\max_{\overline{\Omega}} |u - g_\Omega w| \le C\mathrm{mosc}_\Omega^{(n)} g \le C\mathrm{mosc}_\Omega^{(n)} f \le C\varepsilon. \tag{6-2}$$

Temporarily assume that u is smooth. Recall the operator $\mathcal{M}u = (\det D^2 u)^{1/n}$ and its linearized operator $L_u v$ defined in (3-1). By the concavity of the operator $\mathcal{M}u$, one obtains

$$\frac{g}{g_\Omega} - 1 = \mathcal{M}\left(\frac{u}{g_\Omega}\right) - \mathcal{M}w \le L_w\left(\frac{u}{g_\Omega} - w\right) \qquad \text{in } \Omega.$$

From the interior C^∞ estimates for (6-1), the linear operator L_w is uniformly elliptic in Ω_α with $0 < \alpha < 1$. Let $v = u - g_\Omega w$. By one-sided $W^{2,\delta}$ estimates in [C-C], there exists $0 < \delta_0 < 1$ such that

$$\begin{aligned} &|\Omega_\alpha \backslash \{x_0 \in \Omega_\alpha :\ v = P_{M_0} \text{ at } x_0 \text{ and } v \le P_{M_0} \text{ in } \Omega\}| \\ \le\, &C(\|v\|_{L^\infty(\Omega)} + \|g - g_\Omega\|_{L^n(\Omega)})^{\delta_0} M_0^{-\delta_0}, \end{aligned} \tag{6-3}$$

where $P_{M_0}(x) = M_0|x - x_0|^2 + \ell(x)$ and $\ell(x)$ is an affine function. Through an approximation, (6-3) still holds for the weak solution u. By (6-2) and selecting a suitable M_0, and since $w \in C^2$, there exists $M_1 > 0$ such that

$$|\Omega_\alpha \backslash \{x_0 \in \Omega_\alpha : u = P_{M_1} \text{ at } x_0 \text{ and } u \le P_{M_1} \text{ in } \Omega\}| \le C\varepsilon^{\delta_0} M_0^{-\delta_0} \le \varepsilon^{\delta_0}|\Omega_\alpha|.$$

To finish the proof, it remains to show that if u can be touched from above by P_{M_1} at $x_0 \in \Omega_\alpha$, then $x_0 \in A_\sigma$ for some $\sigma > 0$. Let $u(x) \le M_1|x - x_0|^2 + \ell_{x_0}(x)$ in Ω, where $x_0 \in \Omega_\alpha$ and $\ell_{x_0}(x) = u(x_0) + Du(x_0)(x - x_0)$. Together with Lemma 2.1, there exists $\eta_\alpha > 0$ such that for $h \le \eta_\alpha$

$$B_{\sqrt{h/M_1}}(x_0) \subset S_h(x_0) \subset \Omega_{\frac{1+\alpha}{2}}. \tag{6-4}$$

From the theory of the Monge-Ampère equation, $|S_h(x_0)| \approx h^{n/2}$ for $h \le \eta_\alpha$. Therefore by (6-4), it implies that $diam(S_h(x_0)) \le C(M_1^{n-1}h)^{1/2}$ and $S_h(x_0) \subset B_{C\sqrt{M_1^{n-1}h}}(x_0)$. It follows that

$$u(x) \ge \frac{C}{M_1^{n-1}}|x - x_0|^2 + \ell_{x_0}(x), \quad \text{if } x \in S_{\eta_\alpha}(x_0).$$

We conclude that u is touched by P_σ from below in Ω if $\sigma \le \min\{\frac{C}{M_1^{n-1}}, \frac{\eta_\alpha}{4}\}$. Therefore, $x_0 \in A_\sigma$ and Lemma 6.1 is proved. $\square$

Lemma 6.2. *Let u be a weak solution to (1-1) in Ω satisfying $u = 0$ on $\partial\Omega$. Suppose that $B_{\alpha_n} \subset \Omega \subset B_1$ is convex, $\lambda \le f \le \Lambda$, and $mosc_S^{(n)} f \le \varepsilon$ for any section $S = S_h(x_0) \subset\subset \Omega$. For $0 < \alpha_0 < 1$, let η_{α_0} is the constant by Lemma 2.1 such that (2-1) holds and $S_h(x_0) \subset \Omega_{\frac{\alpha_0+1}{2}}$ for $x_0 \in \Omega_{\alpha_0}$ and $h \le \eta_{\alpha_0}$. Then there exists constants C_0 and δ_0 such that for $x_0 \in \Omega_{\alpha_0}$ and $C_0\beta^{-1} \le h \le \frac{1}{2}\eta_{\alpha_0}$*

$$|S_h(x_0) \setminus A_{\beta^{-1}}| \le \varepsilon^{\delta_0}|S_h(x_0)|,$$

where C_0 and δ_0 $(= \delta_0(\frac{1}{2})$ in Lemma 6.1) depend only on λ, Λ, and n.

Proof. The idea is to use a normalization and Lemma 6.1. For $h \le \eta_{\alpha_0}$, let $Tx = A(x - x_0) + y_0$ be the affine transformation such that $B_{1/n} \subset TS_h(x_0) \subset B_1$. Set $u^*(y) = |\det A|^{2/n}\left[(u - \ell_{x_0})(T^{-1}y) - h\right]$. We have

$$\det D^2u^* = f^*(y) = f(T^{-1}y) \quad \text{in } S^* = TS_h(x_0),$$

and $u^* = 0$ on ∂S^*. Obviously

$$\fint_{S^*} |f^* - (f^*)_{S^*}|^n \, dy = \fint_{S_h(x_0)} |f - f_{S_h(x_0)}|^n \, dx \le \varepsilon^n.$$

Applying Lemma 6.1 with $\alpha = \frac{1}{2}$ to u^* in S^*, we obtain

$$|S^*_{1/2} \setminus A_\sigma(u^*)| \le \varepsilon^{\delta_0}|S^*_{1/2}|, \tag{6-5}$$

where $S^*_{1/2} = \{y \in S^* : u^* < \frac{1}{2}\min_{S^*} u^*\} = T(S_{h/2}(x_0))$.

We now show that there exists a constant $C_0 > 0$ such that

$$T^{-1}(S^*_{1/2} \cap A_\sigma(u^*)) \subset S_{h/2}(x_0) \cap A_{C_0^{-1}h}(u). \tag{6-6}$$

Let $y_1 \in S^*_{1/2} \cap A_\sigma(u^*)$ and $x_1 = T^{-1}y_1 \in S_{h/2}(x_0)$. Then

$$u^*(y) - \ell^*(y) \ge \sigma|y - y_1|^2 \quad \text{for } y \in S^*,$$

where ℓ^* is the supporting function of u^* at $y = y_1$. Recall $|\det A|^{2/n}h = |\min_{S^*} u^*| \approx$ *const*. It follows that

$$u(x) - \ell_{x_1}(x) \ge \sigma Ch|Tx - Tx_1|^2, \quad \text{for } x \in S_h(x_0).$$

Since $diam(S_h(x_0)) \le 2$ and $TS_h(x_0) = S^*$, T is a dilation and $|Tx - Tx_1|^2 \ge C|x - x_1|^2$. We obtain

$$u(x) - \ell_{x_1}(x) \ge Ch|x - x_1|^2, \quad \text{for } x \in S_h(x_0).$$

By $y_1 \in S^*_{1/2}$ and Lemma 2.1, there exists $\eta > 0$ such that $S_\eta(y_1, u^*) \subset\subset S^*$. Obviously

$$S_{K\eta h}(x_1, u) = T^{-1} S_\eta(y_1, u^*) \subset T^{-1} S^* = S_h(x_0),$$

where $K = |\det A|^{-2/n} h^{-1} \approx const.$ For $x \in \Omega \setminus S_h(x_0)$

$$u(x) - \ell_{x_1}(x) \geq K\eta h \geq Ch|x - x_1|^2.$$

Therefore, $x_1 \in A_{C_0^{-1}h}$ and (6-6) is proved. Lemma 6.2 follows immediately from (6-5) and (6-6). $\square$

Fix $0 < \alpha_0 < 1$. Let D_β denote the set

$$D_\beta = \{x_0 \in \Omega_{\alpha_0} : S_h(x_0) \subset B_{\beta\sqrt{h}}(x_0) \text{ for } h \leq \eta_{\alpha_0}\}.$$

By [Gu, Lemma 6.2.2], $D_\beta = \Omega_{\alpha_0} \cap A_{\beta^{-2}}$ if $\beta \geq C_{\alpha_0} = 2/\sqrt{\eta_{\alpha_0}}$.

The following theorem concerns the power decay of the distribution function of the Hessian of solutions.

Theorem 6.3. *Let u be a weak solution to (1-1) and $u = 0$ on $\partial\Omega$. Suppose that $B_{\alpha_n} \subset \Omega \subset B_1$ is convex, $\lambda \leq f \leq \Lambda$, and $mosc^{(n)}_S f \leq \varepsilon$ with small ε for any section $S = S_h(x_0) \subset\subset \Omega$. Then there exist constants $M = M(\alpha_0)$, $C_1 = C_1(\alpha_0)$, p_0 such that for all $\beta \geq C_1$, $0 < \alpha \leq \alpha_0$, and $0 < \tau \leq \alpha - (M\beta)^{-p_0}$*

$$|\Omega_\tau \setminus D_{M\beta}| \leq 2\varepsilon^{\delta_0/2} |\Omega_\alpha \setminus D_\beta|,$$

where δ_0 is as in Lemma 6.2.

Proof. The proof is similar to Theorem 6.3.2 in [Gu]. It is easy to see that the set $\mathcal{O} = \Omega_\tau \setminus D_{M\beta}$ is open and $\mathcal{O} = \Omega_\tau \setminus A_{(M\beta)^{-2}}$, for $M\beta \geq C_{\alpha_0}$ and $\tau < \alpha_0$. Consequently

$$S_h(x_0) \cap \mathcal{O} \subset S_h(x_0) \setminus A_{(M\beta)^{-2}}.$$

By Lemma 6.2, we obtain

$$\frac{|S_h(x_0) \cap \mathcal{O}|}{|S_h(x_0)|} \leq \frac{|S_h(x_0) \setminus A_{(M\beta)^{-2}}|}{|S_h(x_0)|} \leq \varepsilon^{\delta_0}, \tag{6-7}$$

for $C_0(M\beta)^{-2} \leq h \leq \frac{1}{2}\eta_{\alpha_0}$ and $x_0 \in \Omega_{\alpha_0}$.

Since $\mathcal{O}$ is open, we have

$$\lim_{h \to 0} \frac{|S_h(x_0) \cap \mathcal{O}|}{|S_h(x_0)|} = 1, \quad \text{for } x_0 \in \mathcal{O}.$$

By (6-7), for $x_0 \in \mathcal{O}$ we choose h_{x_0}, the largest $h \leq C_0(M\beta)^{-2}$ such that

$$\frac{|S_h(x_0) \cap \mathcal{O}|}{|S_h(x_0)|} = 4\varepsilon^{\delta_0}.$$

Applying the Calderón-Zygmund decomposition ([Gu, Theorem 6.3.3]), we obtain a family of sections $\{S_{h_k}(x_k)\}_{k=1}^{\infty}$, $x_k \in \mathcal{O}$ with $h_k \leq C_0(M\beta)^{-2}$ such that $\mathcal{O} \subset \cup_{k=1}^{\infty} S_{h_k}(x_k)$, $|\mathcal{O}| \leq 2\varepsilon^{\delta_0/2}| \cup_{k=1}^{\infty} S_{h_k}(x_k)|$, and

$$\frac{|S_{h_k}(x_k) \cap \mathcal{O}|}{|S_{h_k}(x_k)|} = 4\varepsilon^{\delta_0}. \tag{6-8}$$

To finish the proof of Theorem 6.3, it suffices to show that

$$S_{h_k}(x_k) \subset \Omega_\alpha \setminus D_\beta. \tag{6-9}$$

By the Alexandrov estimate, $dist(\Omega_\tau, \partial\Omega_\alpha) \geq C(\alpha-\tau)^n$. It follows from Lemma 2.1 that

$$S_{h_k}(x_k) \subset \Omega_{\tau+(M\beta)^{-p_0}} \subset \Omega_\alpha,$$

since $\beta \geq C_1$ and $Ch_k^q \leq C(M\beta)^{-2q} \leq (M\beta)^{-p_0 n}$ for $p_0 = q/n$. We now use a contradiction argument to prove (6-9). Suppose there exists $x_0 \in S_{h_k}(x_k) \cap D_\beta$. By the engulfing property

$$S_{2h_k}(x_k) \subset S_{\theta 2h_k}(x_0) \subset B_{\beta\sqrt{\theta 2h_k}}(x_0). \tag{6-10}$$

Let $Tx = A(x-x_k) + y_k$ be the affine transformation such that $B_{1/n} \subset TS_{2h_k}(x_k) \subset B_1$. Set $u^*(y) = |\det A|^{2/n}\big[(u-\ell_{x_k})(T^{-1}y) - 2h_k\big]$. Then we have

$$\det D^2u^* = f^*(y) = f(T^{-1}y) \qquad \text{in } S_k^* = TS_{2h_k}(x_k),$$

and $\operatorname{mosc}^{(n)}_{S_k^*} f^* = \operatorname{mosc}^{(n)}_{S_{2h_k}(x_k)} f$.

Applying Lemma 6.1 with $\alpha = \frac{1}{2}$ to u^* in S_k^*, we obtain

$$|(S_k^*)_{\frac{1}{2}} \setminus A_\sigma(u^*)| \leq \varepsilon^{\delta_0}|(S_k^*)_{\frac{1}{2}}|,$$

where $(S_k^*)_{\frac{1}{2}} = T(S_{h_k}(x_k))$. We claim that for large M

$$T^{-1}\big((S_k^*)_{\frac{1}{2}} \cap A_\sigma(u^*)\big) \subset \Omega_\alpha \cap D_{M\beta}. \tag{6-11}$$

Let $y_1 \in (S_k^*)_{\frac{1}{2}} \cap A_\sigma(u^*)$ and $x_1 = T^{-1}y_1 \in S_{h_k}(x_k)$. Since $S_h(y_1, u^*) \subset B_{\sqrt{h/\sigma}}(y_1)$, we have for $h \leq C$

$$T^{-1}(S_h(y_1, u^*)) \subset T^{-1}(B_{\sqrt{h/\sigma}}(y_1)).$$

From (6-10), A dilates by at least $(C\beta\sqrt{h_k})^{-1}$ and A^{-1} contracts by at least $C\beta\sqrt{h_k}$. It follows that for $h \leq C$

$$S_{2Kh_kh}(x_1, u) \subset T^{-1}(B_{\sqrt{h/\sigma}}(y_1)) \subset B_{C\beta\sqrt{h_k}\sqrt{h/\sigma}}(x_1),$$

where $K^{-1} = |\min_{S_k^*} u^*| \approx \mathit{const}$. Therefore, $S_h(x_1) \subset B_{C\beta\sqrt{h}}(x_1)$ for $h \le Ch_k$.

Obviously x_0, $x_1 \in S_h(x_k)$ for $h \ge h_k$. By the engulfing property, for $h_k \le h \le \eta_{\alpha_0}/\theta$

$$S_h(x_1) \subset S_{\theta h}(x_0) \subset B_{\beta\sqrt{\theta h}}(x_0) \subset B_{2\beta\sqrt{\theta h}}(x_1).$$

Thus $x_1 \in D_{M\beta}$ for large M and (6-11) follows.

It is easy to see from (6-11) that

$$S_{h_k}(x_k) \cap \mathcal{O} \subset S_{h_k}(x_k) \setminus (\Omega_\alpha \cap D_{M\beta}) \subset T^{-1}((S_k^*)_{\frac{1}{2}} \setminus A_\sigma(u^*)).$$

It implies that

$$\frac{|S_{h_k}(x_k) \cap \mathcal{O}|}{|S_{h_k}(x_k)|} \le \frac{|(S_k^*)_{\frac{1}{2}} \setminus A_\sigma(u^*)|}{|(S_k^*)_{\frac{1}{2}}|} \le \varepsilon^{\delta_0},$$

which contradicts with (6-8). Thus, (6-9) is proved and Theorem 6.3 follows. □

Proof of Theorem A. We first establish the L^p estimates under the assumptions of Theorem 6.3. Let $\alpha_0 = \frac{3}{4}$ and $\alpha_{k+1} = \alpha_k - M^{-(k+2)p_0}$ for $k \ge 0$. By Theorem 6.3, we have $|\Omega_{\alpha_{k+1}} \setminus D_{M^{k+2}}| \le 2\varepsilon^{\delta_0/2}|\Omega_{\alpha_k} \setminus D_{M^{k+1}}|$ for $k \ge 0$. Obviously $\alpha_k \ge \frac{1}{2}$ if M is large. Therefore

$$|\Omega_{1/2} \setminus D_{M^{k+1}}| \le \left(2\varepsilon^{\delta_0/2}\right)^k |\Omega|, \qquad \text{for } k \ge 0. \tag{6-12}$$

By $D_\beta = \Omega_{3/4} \cap A_{\beta^{-2}}$ and [Gu, Lemma 6.2.1], for $x_0 \in D_\beta$, we have $u(x) - \ell_{x_0}(x) \le C\beta^{2(n-1)}|x - x_0|^2$ in Ω if β is large. Let $\Delta_{he}^2 u(x) = [u(x+he) + u(x-he) - 2u(x)]/h^2$. Then on D_β, $\Delta_{he}^2 u \le C\beta^{2(n-1)}$ for small h and $|e| = 1$, and $D_{ee}u \le C\beta^{2(n-1)}$ if $D^2u(x)$ exists. It follows from (6-12) that if $2\varepsilon^{\delta_0/2}M^{2p(n-1)} < 1$, then $\|\Delta_{he}^2 u\|_{L^p(\Omega_{1/2})} \le C$ and

$$\|D^2 u\|_{L^p(\Omega_{1/2})} \le C. \tag{6-13}$$

To prove Theorem A(i), given $\Omega' \subset\subset \Omega$, $x_0 \in \Omega'$, and $p \ge 1$, by Lemma 2.1 and $f \in \mathrm{VMO}_{loc}(\Omega, u)$, choose ε and h_0 small enough such that $2\varepsilon^{\delta_0/2}M^{2p(n-1)} < 1$ and $\operatorname{mosc}_S^{(n)} f \le \varepsilon$ for any section $S = S_h(x_1) \subset S_{2h_0}(x_0)$. Let $Tx = A(x - x_0) + y_0$ be the transform such that $B_{1/n} \subset TS_{2h_0}(x_0) \subset B_1$. Set $u^*(y) = |\det A|^{2/n}\big[(u - \ell_{x_0})(T^{-1}y) - 2h_0\big]$. Obviously

$$\det D^2 u^* = f^*(y) = f(T^{-1}y) \qquad \text{in } S^* = TS_{2h_0}(x_0),$$

and $u^* = 0$ on ∂S^*. Given $y_1 = Tx_1$ and $S_t(y_1, u^*) \subset\subset S^*$, one can verify $S_t(y_1, u^*) = TS_{K_1 t}(x_1, u)$ and $\operatorname{mosc}^{(n)}_{S_t(y_1,u^*)} f^* = \operatorname{mosc}^{(n)}_{S_{K_1 t}(x_1,u)} f \le \varepsilon$, where $K_1 = |\det A|^{-2/n}$. By (6-13), $\|D^2 u^*\|_{L^p(S^*_{1/2})} \le C$, where $S^*_{1/2} = TS_{h_0}(x_0, u)$. Because $B_{C_1 h_0}(x_0) \subset S_{2h_0}(x_0, u) \subset B_{C_2 h_0^q}(x_0)$, we have $TB_{C_1 h_0}(x_0) \subset B_1$. It implies $\|A\| \le Ch_0^{-1}$. Similarly, $\|A^{-1}\| \le$

Ch_0^q. From the L^p estimates of D^2u^* and by a standard covering argument, we obtain $\|D^2u\|_{L^p(\Omega')} \le C$.

To prove Theorem A(ii), for $x_0 \in \Omega_{1/2}$ and small h_0, we have $\mathrm{mosc}_S^{(n)} f \le C_f \psi(C_2 h_0^q) = \varepsilon$ for any $S = S_h(x_1) \subset S_{2h_0}(x_0)$, where $C_f = [f]_{\mathrm{VMO}^\psi(\Omega_{3/4},u)}$. Let u^* and S^* be as in the proof of Theorem A(i). Applying (6-12) to u^* in S^* and noticing $|D^2u^*| \le C\beta^{2(n-1)}$ on D_β, we obtain

$$|\{y \in S^*_{1/2} : |D^2u^*(y)| > CM^{2(k+1)(n-1)}\}| \le \left(2\varepsilon^{\delta_0/2}\right)^k |S^*|, \qquad \text{for } k \ge 0.$$

Since $\eta^k = \left(M_1^k\right)^{\frac{\log \eta}{\log M_1}}$, it yields

$$|\{y \in S^*_{1/2} : |D^2u^*(y)| > \beta\}| \le C\beta^{C_3 \log \varepsilon}, \qquad \text{if } \beta \ge C_4.$$

By the definition of u^*, $|D^2u^*(y)| \le \beta$ implies $|D^2u(T^{-1}y)| \le Ch_0^{-1}\beta$. Therefore

$$|\{x \in S_{h_0}(x_0) : |D^2u(x)| > Ch_0^{-1}\beta\}| \le C\beta^{C_3 \log \varepsilon}|S_{h_0}(x_0)|,$$

for $\beta \ge C_4$. Choose $h_0 = 1/\beta$. One obtains for large β

$$|\{x \in S_{1/\beta}(x_0) : |D^2u(x)| > C\beta^2\}| \le C\beta^{C_3 \log \psi(C_2\beta^{-q})}\beta^{\overline{C}_f - n/2},$$

where $\overline{C}_f = C_3 \log C_f$. Cover $\Omega_{1/2}$ by a finite number of sections $\{S_{1/\beta}(x_i)\}_{i=1}^m$ with $m \le C\beta^n$ and $x_i \in \Omega_{1/2}$. We have for $\beta \ge C_5$

$$|\{x \in \Omega_{1/2} : |D^2u(x)| > C\beta^2\}| \le C\beta^{C_3 \log \psi(C_2\beta^{-q})}\beta^{\overline{C}_f + n/2}. \tag{6-14}$$

Clearly $\log \psi(C_2\beta^{-q/2}) \le C \log \psi(\beta^{-1})$ from the almost monotonicity of $\frac{\log \psi(t)}{\log t}$. By (6-14)

$$|\{x \in \Omega_{1/2} : |D^2u(x)| > \beta\}| \le C\beta^{C_3 \log \psi(\beta^{-1})}\beta^{\overline{C}_f}, \qquad \text{if } \beta \ge C_5.$$

It is easy to verify that if $2^k \ge C_5$

$$\int_{\Omega_{1/2} \cap \{2^k \le |D^2u| < 2^{k+1}\}} \Psi(|D^2u|)\, dx \le C \left(2^k\right)^{\frac{1}{2} C_3 \log \psi(2^{-k})} 2^{k\overline{C}_f},$$

where $\Psi(t) = t^{\sigma C_3 \log(\frac{1}{\psi(1/t)})}$ for $t \ge 1$ and $\Psi(t) = 1$ for $0 \le t \le 1$. Therefore

$$\int_{\Omega_{1/2}} \Psi(|D^2u|)\, dx \le C.$$

Finally, to show $\Psi(|D^2u|) \in L^1_{loc}(\Omega)$, we notice that $diam(T^{-1}S) \le \|A^{-1}\| diam(S)$ for $S = S_t(y_1, u^*)$, and $\Psi(\eta t) \ge t^{C_3 K \log \frac{1}{\psi(1/t)}}$ for $0 < \eta < 1$ and $t \ge \eta^{-2}$, where K depends only on ψ. Then $\Psi(|D^2u|) \in L^1_{loc}(\Omega)$ follows from a standard covering argument. The proof of Theorem A is finished. $\square$

Remark 6.4. Without the assumption in Theorem A(ii) that $\frac{\log \psi(t)}{\log t}$ is almost increasing, one can still show $\Psi_1(|D^2u|) \in L^1(\Omega_{1/2})$, where $\Psi_1(t) = t^{C_3 \log \frac{1}{\psi(C_2 t^{-q/2})}}$ for $t \ge 1$. Furthermore, by (5-4), $q = \frac{1}{2} - \eta$ and η can be made arbitrarily small.

§7 Estimates of the mean oscillation

In this section, we establish the estimates of the mean oscillation for the second derivatives of solutions to (1-1) in Theorem B. These estimates were established in [H] under the stronger assumption that $f(x) \in C(\overline{\Omega})$ with the modulus of continuity ψ. We use Lemma 3.2 and follow the method in [H] to prove Theorem B. Theorem B(ii) gives Caffarelli's $C^{2,\alpha}$ estimates if $\psi(t) = t^\alpha$, and it is better than C^2 regularity in [W], [Ji-W] if ψ is Dini continuous and $\frac{t^\gamma}{\psi(t)}$ is almost increasing, because the functions in $\mathrm{VMO}^\psi(\Omega)$ have the modulus of continuity $C\int_0^t \frac{\psi(r)}{r}$ and f is only required to be in $\mathrm{VMO}^\psi_{loc}(\Omega, u)$.

We firstly prove the following estimates for normalized solutions.

Theorem 7.1. *Let u be a weak solution to (1-1) in Ω satisfying $u = 0$ on $\partial\Omega$ and $D^2u \in L^1(\Omega)$. Assume that $B_{\alpha_n} \subset \Omega \subset B_1$ and $\lambda \le f = g^n \le \Lambda$. Let $u(x_0) = \min_\Omega u$. Suppose that $\mathrm{mosc}^{(n)}_\Omega f \le \omega(2) \ll \mu$ for some small μ and $\mathrm{mosc}^{(n)}_S f \le \omega(diam(S))$ for $S = S_{\mu^k}(x_0, u)$, $k \ge 1$, where $\omega(0) = 0$, ω is increasing and $r^\gamma/\omega(r)$ is almost increasing in $(0, 2]$ for some $0 < \gamma < 1$. Then there exist $0 < \delta < 1$ and small $r_0 > 0$ such that for $0 < r \le r_0$*

$$\inf_{M \in \mathcal{S}} ⨍_{B_r(x_0)} |D^2u - M|^\delta \le C\,\widetilde{\omega}^\delta(r)\left[1 + \int_\Omega |D^2u|^\delta\right],$$

where $\widetilde{\omega}(r) = e^{C_1\Phi(r)}[r^\gamma + \omega(r)]$, $\Phi(r) = \int_r^2 \frac{\omega(t)}{t}\,dt$, and $\mathcal{S}$ denotes the space of real $n \times n$ symmetric matrices.

Proof. Obviously, $\mathrm{mosc}^{(n)}_S g \le C\mathrm{mosc}^{(n)}_S f$. Let A_k and $\mathcal{T}_k$ be the linear transforms given in Lemma 5.2. Consider $u^*(y) = \mu^{-k}[u(x_0 + \mu^{k/2}\mathcal{T}_k^{-1}y) - (\min_\Omega u + \mu^k)]$ for $y \in S^* = \mu^{-k/2}\mathcal{T}_k[S_{\mu^k}(x_0, u) - x_0]$. It is easy to verify that

$$\det D^2u^* = f^* = (g^*)^n \qquad \text{in } S^*,$$

and $u^* = 0$ on ∂S^*, where $f^*(y) = f(x_0 + \mu^{k/2}\mathcal{T}_k^{-1}y)$. By Lemma 5.2

$$\mathrm{mosc}^{(n)}_{S^*} f^* = \mathrm{mosc}^{(n)}_{S_{\mu^k}(x_0,u)} f \le \omega(Ce^{C_\mu\Phi(\sqrt{\mu}^k)}\sqrt{\mu}^k),$$

where $\Phi(r) = \int_r^2 \frac{\omega(t)}{t}\,dt$.

Let w be the smooth solution to

$$\det D^2w = 1 \qquad \text{in } S^*,$$

satisfying $w = 0$ on ∂S^*. By Lemma 3.2

$$|u^* - (g^*)_{S^*}w| \le C\,\mathrm{mosc}^{(n)}_{S^*} g^* \le C\,\omega(e^{C_\mu\Phi(\sqrt{\mu}^k)}\sqrt{\mu}^k), \quad \text{in } S^*. \tag{7-1}$$

Recall the following Campanato inequality (see (3.2) in [H])

$$\inf_{M \in \mathcal{S}} \int_{B_\rho} |D^2w - M|^p \le C_p\left(\frac{\rho}{R}\right)^{n+p} \inf_{M \in \mathcal{S}} \int_{B_R} |D^2w - M|^p, \tag{7-2}$$

for $p > 0$, $0 < \rho < R \le R_0$, where R_0 is some constant such that $B_{2R_0} \subset S^*$. Let $v = u^* - (g^*)_{S^*} w$. It follows for $0 < \rho \le R_0$, $0 < p \le 1$

$$
\begin{aligned}
&\inf_{M\in\mathcal{S}} \int_{B_\rho} |D^2 u^* - M|^p \\
&\le C_p \rho^{n+p} \inf_{M\in\mathcal{S}} \int_{B_{R_0}} |(g^*)_{S^*} D^2 w - M|^p + \int_{B_\rho} |D^2 v|^p \\
&\le C_p \rho^{n+p} \inf_{M\in\mathcal{S}} \int_{B_{R_0}} |D^2 u^* - M|^p + C_p \int_{B_{R_0}} |D^2 v|^p.
\end{aligned}
\tag{7-3}
$$

To estimate the integral of $|D^2 v|^p$, we temporarily assume that u^* is smooth. Recall $\mathcal{M}u = (\det D^2 u)^{1/n}$ and its linearized operator $L_u v$ given in (3-1). Since $\mathcal{M}$ is concave, we obtain

$$
L_{u^*} v \le \mathcal{M}u^* - \mathcal{M}((g^*)_{S^*} w) = g^* - (g^*)_{S^*} \le L_{(g^*)_{S^*} w} v, \quad \text{in } S^*.
$$

Because $L_{(g^*)_{S^*} w}$ is uniformly elliptic in $B_{\frac{3}{2}R_0}$, by one-sided $W^{2,\delta}$ estimates in [C-C, Lemma 7.8] and (7-1), there exists $0 < \delta_1 < 1$ such that

$$
\left(\fint_{B_{R_0}} |(\Delta^2_{he} v)^+|^{\delta_1} \right)^{1/\delta_1} \le C\big(\|v\|_{L^\infty(S^*)} + \operatorname{mosc}^{(n)}_{S^*} g^*\big) \le C \operatorname{mosc}^{(n)}_{S^*} g^*,
\tag{7-4}
$$

where $\Delta^2_{he} v(x) = [v(x+he) + v(x-he) - 2v(x)]/h^2$ for $|e| = 1$, small $h > 0$. On the other hand, $\hat{L}_{u^*} v = \operatorname{tr}\big(\det D^2 u^* (D^2 u^*)^{-1} D^2 v\big) \le C_1\big(g^* - (g^*)_{S^*}\big)$ in S^*. The $W^{2,\delta}$ estimates with small δ for the equation $\hat{L}_{u^*} v = g$ were established in [Gu-To] under the assumption $\lambda \le \det D^2 u^* \le \Lambda$. A straightforward modification of their proof gives rise to one-sided $W^{2,\delta}$ estimates for supersolutions. More precisely, there exists $0 < \delta_2 < 1$ such that

$$
\left(\fint_{B_{R_0}} |(\Delta^2_{he} v)^-|^{\delta_2} \right)^{1/\delta_2} \le C\big(\|v\|_{L^\infty(S^*)} + \operatorname{mosc}^{(n)}_{S^*} g^*\big) \le C \operatorname{mosc}^{(n)}_{S^*} g^*.
\tag{7-5}
$$

Through an approximation, (7-4) and (7-5) also holds for the weak solution u^*. Since $D^2 u^* \in L^1(S^*)$, $\Delta^2_{he} v$ converges to $D_{ee} v$ in $L^1_{loc}(S^*)$ and hence $L^p_{loc}(S^*)$ for $0 < p < 1$. Choose $p = \delta = \min\{\delta_1, \delta_2\}$ in (7-3). By (7-3)–(7-5), we obtain for $0 < \rho \le R_0$

$$
\inf_{M\in\mathcal{S}} \int_{B_\rho} |D^2 u^* - M|^\delta \le C \rho^{n+\delta} \inf_{M\in\mathcal{S}} \int_{B_{R_0}} |D^2 u^* - M|^\delta + C\omega^\delta\big(e^{C_\mu \Phi(\sqrt{\mu}^k)} \sqrt{\mu}^k\big).
$$

It is easy to verify that $S_\mu(0, u^*) = \mu^{-k/2} \mathcal{T}_k [S_{\mu^{k+1}}(x_0, u) - x_0]$. By Lemma 5.2, $B_{C^{-1}} \subset \mu^{-(k+1)/2} A_{k+1} \mathcal{T}_k [S_{\mu^{k+1}}(u, x_0) - x_0] \subset B_C$. Therefore $B_{C^{-1}\sqrt{\mu}} \subset S_\mu(0, u^*) \subset B_{C\sqrt{\mu}}$. We have

$$
\inf_{M\in\mathcal{S}} \int_{S_\mu(0,u^*)} |D^2 u^* - M|^\delta \le C \sqrt{\mu}^{n+\delta} \inf_{M\in\mathcal{S}} \int_{S^*} |D^2 u^* - M|^\delta + C\omega^\delta\big(e^{C_\mu \Phi(\sqrt{\mu}^k)} \sqrt{\mu}^k\big).
$$

By the change of variables, it follows that

$$
\begin{aligned}
&\inf_{M\in\mathcal{S}}\int_{S_{\mu^{k+1}}(x_0,u)}|(\mathcal{T}_k^{-1})^t(D^2u-M)\mathcal{T}_k^{-1}|^\delta\\
&\quad\le C\sqrt{\mu}^{n+\delta}\inf_{M\in\mathcal{S}}\int_{S_{\mu^k}(x_0,u)}|(\mathcal{T}_k^{-1})^t(D^2u-M)\mathcal{T}_k^{-1}|^\delta\\
&\qquad+C\sqrt{\mu}^{kn}\omega^\delta(e^{C_\mu\Phi(\sqrt{\mu}^k)}\sqrt{\mu}^k).
\end{aligned}\tag{7-6}
$$

For simplicity, let $\bar\mu=\sqrt{\mu}$ and

$$
b_k=\inf_{M\in\mathcal{S}}\int_{S_{\mu^k}(x_0,u)}|(\mathcal{T}_k^{-1})^t(D^2u-M)\mathcal{T}_k^{-1}|^\delta.
$$

Similar to the proof on [H, page 613], if μ is small enough, we have for $k\ge 1$

$$
\begin{aligned}
b_{k+1}&\le\bar\mu^{k(n+\delta\beta)}b_1+C\bar\mu^{kn}h^\delta(\bar\mu^k)\sum_{i=1}^k\bar\mu^{(k-i)\delta(\beta-\gamma)}\\
&\le C(\bar\mu^k)^n[\bar\mu^{k\gamma}+h(\bar\mu^k)]^\delta(1+b_1),
\end{aligned}
$$

where $\gamma<\beta<1$ and $h(t)=\omega(e^{C_\mu\Phi(t)}t)$. Since $t^\gamma/h(t)$ is almost increasing by [H], we have $b_k\le\overline{C}_\mu(\bar\mu^k)^n[\bar\mu^{k\gamma}+h(\bar\mu^k)]^\delta(1+b_1)$, for $k\ge 2$. By almost increasingness of $t^\gamma/\omega(t)$, $h(t)\le Ce^{\gamma C_\mu\Phi(t)}\omega(t)$. It follows that for $k\ge 2$

$$
\begin{aligned}
\inf_{M\in\mathcal{S}}\int_{S_{\mu^k}(x_0,u)}|D^2u-M|^\delta&\le C\|\mathcal{T}_k\|^{2\delta}b_k\\
&\le\overline{C}_\mu\sqrt{\mu}^{kn}\left[e^{C_\mu\Phi(\sqrt{\mu}^k)}(\sqrt{\mu}^{k\gamma}+\omega(\sqrt{\mu}^k))\right]^\delta(1+b_1).
\end{aligned}\tag{7-7}
$$

Now fix μ small. Recall that $\Phi(t)\le\Phi(\theta t)\le(1+\frac{\log(1/\theta)}{\log 2})\Phi(t)$ and $\Phi(\sqrt{t})\le\Phi(t)\le 3\Phi(\sqrt{t})$, for $0<t,\ \theta\le 1$. From (7-7) and Lemma 5.2, a standard argument yields for $0<\rho\le\mu^2$

$$
\inf_{M\in\mathcal{S}}\int_{S_\rho(x_0,u)}|D^2u-M|^\delta\le C\sqrt{\rho}^n\left[e^{C_1\Phi(\sqrt{\rho})}(\sqrt{\rho}^\gamma+\omega(\sqrt{\rho}))\right]^\delta(1+b_1),\tag{7-8}
$$

and for $0<\rho\le\mu$

$$
B_{C_2^{-1}\sqrt{\rho}e^{-C_1\Phi(\sqrt{\rho})}}(x_0)\subset S_\rho(x_0,u)\subset B_{C_2\sqrt{\rho}e^{C_1\Phi(\sqrt{\rho})}}(x_0).\tag{7-9}
$$

Let $R=(C_2)^2\rho e^{2C_1\Phi(\rho)}$. Then $C_2^{-1}\sqrt{R}e^{-C_1\Phi(\sqrt{R})}\ge\sqrt{\rho}e^{C_1\Phi(\rho)}e^{-C_1\Phi(\rho)}\ge\sqrt{\rho}$. By (7-9), it implies that $B_{\sqrt{\rho}}(x_0)\subset S_R(x_0,u)$. Moreover, since $\Phi(\rho)\le\omega(2)\log(2/\rho)$ and $\omega(2)$

is small, $\rho \le R \le \sqrt{\rho}$ for small ρ. Therefore, from (7-8) and by the properties of Φ, we obtain for small ρ

$$\inf_{M\in\mathcal{S}} \int_{B_{\sqrt{\rho}}(x_0)} |D^2u - M|^\delta \le C\sqrt{R}^n \left[e^{C_1\Phi(\sqrt{R})}(\sqrt{R}^\gamma + \omega(\sqrt{R}))\right]^\delta (1+b_1)$$
$$\le C\sqrt{\rho}^n \left[e^{C_3\Phi(\sqrt{\rho})}(\sqrt{\rho}^\gamma + \omega(\sqrt{\rho}))\right]^\delta (1+b_1).$$

So Theorem 7.1 is proved. □

Proof of Theorem B. For $x_0 \in \Omega_\alpha$, $0<\alpha<1$, and small $h_0>0$, let E be the Fritz John ellipsoid of $S_{h_0}(x_0,u)$ and T be the affine transformation given by $Tx = A(x-x_0)+y_0$ such that $TE = B_1$. Set $u^*(y) = C_A^{-1}\left[(u-\ell_{x_0})(T^{-1}y) - h_0\right]$ with $C_A = |\det A|^{-2/n} \approx h_0$. Obviously

$$\det D^2u^* = f^*(y) \qquad \text{in } S^* = TS_{h_0}(x_0),$$

where $f^*(y) = f(T^{-1}y)$. By Theorem 2.1, choose $h_0 = h_0(\tau,\alpha)$ such that $\|A^{-1}\| \le Ch_0^q \le \tau$ and $S_{h_0}(x_0,u) \subset \Omega_{\frac{1+\alpha}{2}}$. From Theorem A(i), $D^2u \in L^p_{loc}(\Omega)$ and hence $D^2u^* \in L^1(S^*)$. Obviously, $S_{\mu^k}(y_0,u^*) = TS_{C_A\mu^k}(x_0,u)$ for $k\ge 1$. It is easy to verify that $diam(T^{-1}S) \le \tau diam(S)$, $\operatorname{mosc}^{(n)}_{S^*} f^* = \operatorname{mosc}^{(n)}_{S_{h_0}(x_0,u)} f \le C_{f,\alpha}\,\psi(2\tau)$, and $\operatorname{mosc}^{(n)}_{S} f^* \le C_{f,\alpha}\,\psi(\tau diam(S))$ for $S = S_{\mu^k}(y_0,u^*)$, $k\ge 1$, where $C_{f,\alpha} = [f]_{\mathrm{VMO}^\psi(\Omega_\alpha,u)}$. Applying Theorem 7.1 to u^*, one obtains for $0<r\le r_0$

$$\inf_{M\in\mathcal{S}} \int_{B_r(y_0)} |D^2u^* - M|^\delta \le Cr^n\widetilde{\omega}^\delta(r)\left[1 + \int_{S^*}|D^2u^*|^\delta\right], \tag{7-10}$$

where $\widetilde{\omega}(r) = e^{C_1\Phi(r)}[r^\gamma + C_{f,\alpha}\psi(\tau r)]$ and $\Phi(r) = C_{f,\alpha}\int_r^2 \frac{\psi(\tau t)}{t}\,dt$. Note that $\widetilde{\omega}(r) \le C\varphi_\tau(r) = C\psi(r)e^{C_2\int_r^2 \frac{\psi(\tau t)}{t}}$, $TB_{r/\|A\|}(x_0) \subset B_r(y_0)$, and $\psi(r)/r^\gamma$ is almost decreasing. It follows from (7-10) that for $0<\rho = \frac{r}{\|A\|} \le \frac{r_0}{\|A\|}$

$$\inf_{M\in\mathcal{S}} \int_{B_\rho(x_0)} |D^2u - M|^\delta \le C_{\tau,\alpha}\rho^n(\varphi_\tau(\rho))^\delta\left[1 + \int_{\Omega_{\frac{1+\alpha}{2}}}|D^2u|^\delta\right], \tag{7-11}$$

where $C_{\tau,\alpha}$ depends on τ and α.

To finish the proof, we discuss properties of φ_τ. If $\int_0^1 \frac{\psi(t)}{t} < \infty$, $\varphi_\tau(t) \le C\psi(t)$. If $\psi(t) = o\big(\log^{-1}(4/t)\big)$, then for $0<\varepsilon<1$, there exists small τ such that $\psi(\tau t) \le \varepsilon\log^{-1}(4/t)$ for $0<t\le 2$. Therefore, $\varphi_\tau(t) \le C\log^{-(1-C_2\varepsilon)}(4/t)$. If $H(t) = \psi(t)\log(4/t)$ is increasing and $H(0)=0$, one can check that $e^{C_2\int_t^2 \frac{\psi(\tau r)}{r}}\log^{-1}(4/t)$ and $\varphi_\tau(t)$ for small τ are increasing. By [C-H, Corollary 2.3], Theorem B follows from (7-11). □

In the following, we give some non-Dini continuous functions that belong to certain $\mathrm{VMO}^\psi_{loc}(\Omega,u)$ with $\int_0^1 \frac{\psi(t)}{t} < \infty$.

Example 7.2. Let $h(x) = \log^{-1}(\frac{8}{|x|})\big(\log\log\frac{8}{|x|}\big)^{-q}$ for $x \in \overline{B}_1 \subset \mathbb{R}^n$ where $n \geq 2$ and $0 < q < 1$. Obviously, $|h(x) - h(y)| \leq C\psi_1(|x-y|)$ where $\psi_1(t) = \log^{-1}(\frac{8}{t})\big(\log\log\frac{8}{t}\big)^{-q}$ and $h(x)$ is not Dini continuous. Moreover, one can use the Poincaré inequality to show that $h(x) \in \mathrm{VMO}^{\psi_2}(B_1)$ where $\psi_2(t) = \log^{-2}(\frac{8}{t})\big(\log\log\frac{8}{t}\big)^{-q}$.

Consider $f(x) = F(h(x))$, where $F(t)$ is Lipschitz on $[0,\infty)$ and $\lambda \leq F(t) \leq \Lambda$. Obviously, the modulus of continuity of f is of order $\psi_1(t)$ and $f(x) \in \mathrm{VMO}^{\psi_2}(B_1)$. In general, f is non Dini continuous. Let u be the solution to $\det D^2u = f(x)$ in Ω with $u = 0$ on $\partial\Omega$, where $B_{\alpha_n} \subset \Omega \subset B_1$. We claim that $f(x) \in \mathrm{VMO}^{\psi}_{loc}(\Omega, u)$ with $\psi(t) = \log^{-(2-\varepsilon)}(\frac{8}{t})$ for any $0 < \varepsilon < 1$.

To prove the claim, given $0 < \varepsilon < 1$, $\Omega' \subset\subset \Omega$ and $x_0 \in \Omega'$, we use [H, Theorem 2.1] to obtain that $B_{C_1\sqrt{\rho}\log^{-\varepsilon}\frac{8}{\rho}}(x_0) \subset S_\rho(x_0, u) \subset B_{C_2\sqrt{\rho}\log^{\varepsilon}\frac{8}{\rho}}(x_0)$, for $\rho = \rho_0\mu^k$, $k \geq 1$, where μ and ρ_0 are some constants. Note $|S_\rho(x_0,u)| \approx \rho^{n/2}$. Therefore, we have for $\rho = \rho_0\mu^k$

$$\mathrm{mosc}^{(n)}_{S_\rho(x_0,u)} f \leq C\log^{\varepsilon}\frac{8}{\rho}\,\mathrm{mosc}^{(n)}_{B_{diam(S_\rho(x_0,u))}(x_0)} f \leq C\log^{-(2-\varepsilon)}\Big(\frac{8}{diam(S_\rho(x_0,u))}\Big).$$

This proves the claim.

By a similar argument, we have

Remark 7.3. Let $f \in C(\overline{\Omega})$ with the modulus of continuity of order ψ_1 such that $\lambda \leq f \leq \Lambda$ and $f \in \mathrm{VMO}^{\psi_2}(\Omega)$, where $B_{\alpha_n} \subset \Omega \subset B_1$ and $t^\gamma/\psi_1(t)$ is almost increasing for some $0 < \gamma < 1$. Let u be the solution to $\det D^2u = f$ in Ω with $u = 0$ on $\partial\Omega$. Then $f \in \mathrm{VMO}^{\psi_\tau}_{loc}(\Omega, u)$ for any small $\tau > 0$, where $\psi_\tau = \min\{\psi_1, \psi_2\Phi_\tau\}$ and $\Phi_\tau(t) = e^{C\int_t^2 \frac{\psi_1(\tau r)}{r}}$ for some constant $C = C(n, \lambda, \Lambda, \gamma, \psi_1)$.

Acknowledgement. Part of this work was performed while the author was visiting MSRI at Berkeley. The author was grateful to the institution for its hospitality and stimulating atmosphere.

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