THE REFRACTOR PROBLEM IN RESHAPING LIGHT BEAMS

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Contents

1. Introduction

To explain the problem and results in this paper we begin describing the reflector problem. This problem recently received great interest and originates in

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engineering in the study of reflecting surfaces to reshape electromagnetic radiation in a prescribed manner. It can be described as follows. Suppose that Ω, Ω^* are two domains of the unit sphere S^{n-1} in $\mathbb{R}^n$, and light emanates from the point O in an isotropic media with intensity $f(x)$ for $x \in \Omega$. The reflector problem consists in finding a perfectly reflecting surface $\mathcal{R}$, parameterized by a polar expression $\rho(x)x$ for $x \in \Omega$, such that all rays reflected by $\mathcal{R}$ have directions in Ω^*, and the prescribed illumination intensity received in the direction $m \in \Omega^*$ is $f^*(m)$. The PDE governing this problem is an equation of Monge-Ampère type on $\Omega \subset S^{n-1}$, see [GW98], and progress has been made concerning existence, uniqueness, and regularity of solutions. See for instance, [Wan96], [Wan04], [OG03], [CO94], [CGH06], and references therein. The reflector problem in anisotropic media is discussed in [CH06].

The problem considered in this paper concerns *refraction* and appears in connection with the synthesis of refracting surfaces capable of reshaping the intensity of light beam. Mathematically the refractor problem is formulated as follows. Let n_1 and n_2 be the indexes of refraction of two homogeneous and isotropic media I and II, respectively. Suppose that from a point O inside medium I light emanates with intensity $f(x)$ for $x \in \Omega$. We want to construct a refracting surface $\mathcal{R}$ parameterized as $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, separating media I and II, and such that all rays refracted by $\mathcal{R}$ into medium II have directions in Ω^* and the prescribed illumination intensity received in the direction $m \in \Omega^*$ is $f^*(m)$. This implies the existence of a lens refracting light beams in a prescribed way, see Remark 4.8.

To the best of our knowledge there have been no results for this problem, and it is our purpose in this paper to deal with the question of existence and uniqueness of solutions up to dilations. To tackle the problem, we first find surfaces that refract all light rays emanating from a point O into a fix direction; we say that these surfaces have the *uniform refraction property*. Using these surfaces and energy conservation, we then formulate the concept of weak solution to the refractor problem, and next establish existence and uniqueness converting the problem into an optimal mass transfer problem from Ω to Ω^* with a suitable cost function according with the value of $\kappa = n_2/n_1$.

The organization of the paper is as follows. In Section 2, we review the Snell law of refraction in vector form and discuss surfaces having the uniform refraction property. In Section 3 we consider the refractor problem in case $\kappa < 1$. Section 4 discusses the refractor problem when $\kappa > 1$. Finally, in Section 5 we set up the differential equation corresponding to the refractor problem and check the validity of a condition introduced by Ma, Trudinger and Wang in [MTW05].

2. Surfaces with the uniform refraction property

We recall here the physical law of refraction and find the surfaces having the uniform refraction property. It is well known that any paraboloid of revolution reflects all rays of light emitted from its focus and having non-axial direction into light rays in the axial direction. We will show the surfaces having the uniform refraction property are semi-ellipsoids of revolution for $n_2 < n_1$, and a sheet of hyperboloids of revolution of two sheets for $n_2 > n_1$.

2.1. **Snell's law of refraction.** Suppose Γ is a surface in $\mathbb{R}^n$ that separates two media I and II that are homogeneous and isotropic. Let v_1 and v_2 be the velocities of propagation of light in the media I and II respectively. The index of refraction of the medium I is by definition $n_1 = c/v_1$, where c is the velocity of propagation of light in the vacuum, and similarly $n_2 = c/v_2$. If a ray of light* having direction $x \in S^{n-1}$ and traveling through the medium I hits Γ at the point P, then this ray is refracted in the direction $m \in S^{n-1}$ through the medium II according with the Snell law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$, where θ_1 is the angle between x and ν (the angle of incidence), θ_2 the angle between m and ν (the angle of refraction), and ν is the unit normal to Γ at P going towards the medium II. The vectors x, ν and m are coplanar.

*Since the refraction angle depends on the frequency of the radiation, we assume our light ray is monochromatic.

In vector form, the Snell law can be expressed by the fact that the vector $n_2 m - n_1 x$ is parallel to the normal vector ν. If we set $\kappa = n_2/n_1$, then

$$x - \kappa m = \lambda \nu, \tag{2.1}$$

for some $\lambda \in \mathbb{R}$. It can be seen that $\lambda = \cos\theta_1 - \kappa\cos\theta_2$, $\cos\theta_1 = x \cdot \nu > 0$, and $\cos\theta_2 = m \cdot \nu = \sqrt{1 - \kappa^{-2}[1 - (x \cdot \nu)^2]}$.

When $\kappa < 1$, or equivalently $v_1 < v_2$, waves propagate in medium II faster than in medium I, or equivalently, medium I is denser than medium II. In this case the refracted rays tend to bent away from the normal, that is the case for example, when medium I is glass and medium II is air. For this reason, the maximum angle of refraction θ_2 is $\pi/2$ which is achieved when $\sin\theta_1 = n_2/n_1 = \kappa$. So there cannot be refraction when the incidence angle θ_1 is beyond this critical value, that is, we must have $0 \leq \theta_1 \leq \theta_c = \arcsin\kappa$.† It is easy to verify that

$$\theta_2 - \theta_1 = \arcsin(\kappa^{-1}\sin\theta_1) - \theta_1 \tag{2.2}$$

is strictly increasing for $\theta_1 \in [0,\ \theta_c]$, and therefore $0 \leq \theta_2 - \theta_1 \leq \dfrac{\pi}{2} - \theta_c$. We then lead to the following physical constraint:

(2.3) *if $\kappa = n_2/n_1 < 1$ and a ray of direction x through medium I is refracted into medium II in the direction m, then $m \cdot x \geq \kappa$.*

Conversely, given $x, m \in S^{n-1}$ with $x \cdot m \geq \kappa$ and $\kappa < 1$, it follows from (2.2) that there exists a hyperplane refracting any ray through medium I with direction x into a ray of direction m in medium II.

In case $\kappa > 1$, waves propagate in medium I faster than in medium II, and the refracted rays tend to bent towards the normal. By the Snell law, the maximum angle of refraction denoted by θ_c^* is achieved when $\theta_1 = \pi/2$, and $\theta_c^* = \arcsin(1/\kappa)$. Obviously,

$$\theta_1 - \theta_2 = \arcsin(\kappa\sin\theta_2) - \theta_2 \tag{2.4}$$

†If $\theta_1 > \theta_c$, then the phenomenon of total internal reflection occurs.

is strictly increasing for $\theta_2 \in [0, \theta_c^*]$, and $0 \le \theta_1 - \theta_2 \le \dfrac{\pi}{2} - \theta_c^*$. We therefore obtain the following physical constraint for the case $\kappa > 1$:

$$\text{if a ray with direction } x \text{ traveling through medium I} \\ \text{is refracted into a ray in medium II with direction } m, \text{ then } m \cdot x \ge 1/\kappa. \tag{2.5}$$

On the other hand, by (2.4), if $x, m \in S^{n-1}$ with $x \cdot m \ge 1/\kappa$ and $\kappa > 1$, then there exists a hyperplane refracting any ray of direction x through medium I into a ray with direction m in medium II.

We summarize the above discussion on the physical constraints of refraction in the following lemma.

Lemma 2.1. *Let n_1 and n_2 be the indices of refraction of two media I and II, respectively, and $\kappa = n_2/n_1$. Then a light ray in medium I with direction $x \in S^{n-1}$ is refracted by some surface into a light ray with direction $m \in S^{n-1}$ in medium II if and only if $m \cdot x \ge \kappa$, when $\kappa < 1$; and if and only if $m \cdot x \ge 1/\kappa$, when $\kappa > 1$.*

2.2. **Surfaces with the uniform refracting property.** Let $m \in S^{n-1}$ be fixed, and we ask the following: if rays of light emanate from the origin inside medium I, what is the surface Γ, interface of the media I and II, that refracts all these rays into rays parallel to m?

Suppose Γ is parameterized by the polar representation $\rho(x)x$ where $\rho > 0$ and $x \in S^{n-1}$. Consider a curve on Γ given by $r(t) = \rho(x(t))x(t)$ for $x(t) \in S^{n-1}$. According to (2.1), the tangent vector $r'(t)$ to Γ satisfies $r'(t)\cdot(x(t) - \kappa\, m) = 0$. That is, $([\rho(x(t))]'x(t) + \rho(x(t))x'(t))\cdot(x(t) - \kappa\, m) = 0$, which yields $(\rho(x(t))(1 - \kappa m \cdot x(t)))' = 0$. Therefore

$$\rho(x) = \frac{b}{1 - \kappa\, m \cdot x} \tag{2.6}$$

for $x \in S^{n-1}$ and for some $b \in \mathbb{R}^n$. To understand the surface given by (2.6), we distinguish two cases $\kappa < 1$ and $\kappa > 1$.

Let us first consider the case $\kappa < 1$. For $b > 0$, we will see that the surface Γ given by (2.6) is an ellipsoid of revolution about the axis of direction m. Suppose for simplicity that $m = e_n$, the nth-coordinate vector. If $y = (y', y_n) \in \mathbb{R}^n$ is a point on Γ, then $y = \rho(x)x$ with $x = y/|y|$. From (2.6), $|y| - \kappa\, y_n = b$, that is, $|y'|^2 + y_n^2 = (\kappa\, y_n + b)^2$ which yields $|y'|^2 + (1-\kappa^2)y_n^2 - 2\kappa b y_n = b^2$. This surface Γ can be written in the form

$$\frac{|y'|^2}{\left(\dfrac{b}{\sqrt{1-\kappa^2}}\right)^2} + \frac{\left(y_n - \dfrac{\kappa b}{1-\kappa^2}\right)^2}{\left(\dfrac{b}{1-\kappa^2}\right)^2} = 1 \tag{2.7}$$

which is an ellipsoid of revolution about the y_n axis with foci $(0,0)$ and $(0, 2\kappa b/(1-\kappa^2))$. Since $|y| = \kappa y_n + b$ and the physical constraint for refraction (2.3), $\dfrac{y}{|y|} \cdot e_n \geq \kappa$ is equivalent to $y_n \geq \dfrac{\kappa b}{1-\kappa^2}$. That is, for refraction to occur y must be in the upper part of the ellipsoid (2.7); we denote this semi-ellipsoid by $E(e_n, b)$. To verify that $E(e_n, b)$ has the uniform refracting property, that is, it refracts any ray emanating from the origin in the direction e_n, we check that (2.1) holds at each point. Indeed, if $y \in E(e_n, b)$, then $\left(\dfrac{y}{|y|} - \kappa e_n\right)\dfrac{y}{|y|} \geq 1-\kappa > 0$, and $\left(\dfrac{y}{|y|} - \kappa e_n\right) \cdot e_n \geq 0$, and so $\dfrac{y}{|y|} - \kappa e_n$ is an outward normal to $E(e_n, b)$ at y.

Rotating the coordinates, it is easy to see that the surface given by (2.6) with $\kappa < 1$ and $b > 0$ is an ellipsoid of revolution about the axis of direction m with foci 0 and $\dfrac{2\kappa b}{1-\kappa^2}m$. Moreover, the semi-ellipsoid $E(m, b)$ given by

$$E(m, b) = \{\rho(x)x : \rho(x) = \frac{b}{1-\kappa\, m \cdot x},\ x \in S^{n-1},\ x \cdot m \geq \kappa\}, \tag{2.8}$$

has the uniform refracting property, any ray emanating from the origin O is refracted in the direction m.

Now turn to the case $\kappa > 1$. Due to the physical constraint of refraction (2.5), we must have $b < 0$ in (2.6). Define for $b > 0$

$$H(m, b) = \{\rho(x)x : \rho(x) = \frac{b}{\kappa\, m \cdot x - 1},\ x \in S^{n-1},\ x \cdot m \geq 1/\kappa\}. \tag{2.9}$$

We claim that $H(m, b)$ is the sheet with opening in direction m of a hyperboloid of revolution of two sheets about the axis of direction m. To prove the claim, set for simplicity $m = e_n$. If $y = (y', y_n) \in H(e_n, b)$, then $y = \rho(x)x$ with $x = y/|y|$. From (2.9), $\kappa\, y_n - |y| = b$, and therefore $|y'|^2 + y_n^2 = (\kappa\, y_n - b)^2$ which yields $|y'|^2 - (\kappa^2 - 1)\left[\left(y_n - \frac{\kappa b}{\kappa^2-1}\right)^2 - \left(\frac{\kappa b}{\kappa^2-1}\right)^2\right] = b^2$. Thus, any point y on $H(e_n, b)$ satisfies the equation

$$\frac{\left(y_n - \frac{\kappa b}{\kappa^2-1}\right)^2}{\left(\frac{b}{\kappa^2-1}\right)^2} - \frac{|y'|^2}{\left(\frac{b}{\sqrt{\kappa^2-1}}\right)^2} = 1 \tag{2.10}$$

which represents a hyperboloid of revolution of two sheets about the y_n axis with foci $(0, 0)$ and $(0, 2\kappa b/(\kappa^2 - 1))$. Moreover, the upper sheet of this hyperboloid of revolution is given by

$$y_n = \frac{\kappa b}{\kappa^2-1} + \frac{b}{\kappa^2-1}\sqrt{1 + \frac{|y'|^2}{\left(b/\sqrt{\kappa^2-1}\right)^2}}$$

and satisfies $\kappa y_n - b > 0$, and hence has polar equation $\rho(x) = \dfrac{b}{\kappa\, e_n \cdot x - 1}$. Similarly, the lower sheet satisfies $\kappa y_n - b < 0$ and has polar equation $\rho(x) = \dfrac{b}{\kappa\, e_n \cdot x + 1}$. For a general m, by a rotation, we obtain that $H(m, b)$ is the sheet with opening in direction m of a hyperboloid of revolution of two sheets about the axis of direction m with foci $(0, 0)$ and $\dfrac{2\kappa b}{\kappa^2-1} m$.

Notice that the focus $(0, 0)$ is outside the region enclosed by $H(m, b)$ and the focus $\dfrac{2\kappa b}{\kappa^2-1} m$ is inside that region. The vector $\kappa m - \dfrac{y}{|y|}$ is an inward normal to $H(m, b)$ at y, because by (2.9)

$$\begin{aligned}\left(\kappa m - \frac{y}{|y|}\right)\cdot\left(\frac{2\kappa b}{\kappa^2-1} m - y\right) &\geq \frac{2\kappa^2 b}{\kappa^2-1} - \frac{2\kappa b}{\kappa^2-1} - \kappa m \cdot y + |y| \\ &= \frac{2\kappa b}{\kappa+1} - b = \frac{b(\kappa-1)}{\kappa+1} > 0.\end{aligned}$$

Clearly, $\left(\kappa m - \dfrac{y}{|y|}\right)\cdot m \geq \kappa - 1$ and $\left(\kappa m - \dfrac{y}{|y|}\right)\cdot \dfrac{y}{|y|} > 0$. Therefore, $H(m, b)$ satisfies the uniform refraction property.

We remark that one has to use $H(-e_n, b)$ to uniformly refract in the direction $-e_n$, and due to the physical constraint (2.5), the lower sheet of the hyperboloid of equation (2.10) cannot refract in the direction $-e_n$.

From the above discussion, we have proved the following.

Lemma 2.2. *Let n_1 and n_2 be the indexes of refraction of two media I and II, respectively, and $\kappa = n_2/n_1$. Assume that the origin O is inside medium I, and $E(m, b), H(m, b)$ are defined by* (2.8) *and* (2.9)*, respectively. We have:*

(i) *If $\kappa < 1$ and $E(m, b)$ is the interface of media I and II, then $E(m, b)$ refracts all rays emitted from O into rays in medium II with direction m.*
(ii) *If $\kappa > 1$ and $H(m, b)$ separates media I and II, then $H(m, b)$ refracts all rays emitted from O into rays in medium II with direction m.*

Remark 2.3. After finding the surfaces with the uniform refraction property, we learned that in the plane these are discussed in Descartes' Eight Discourse on Optics [Des01, pp. 127-149] and applied to the design of lenses. If $\kappa < 1$ and the ellipse $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, with $a > b$, is filled with a material having refraction index n_1 and the outside of the ellipse is filled with a material having refraction index n_2, then all rays emanating from one focus are refracted by the half of the ellipse closed to the other focus into rays parallel to the x-axis if the eccentricity $e = \sqrt{1-(b/a)^2} = \kappa$. Similarly, if $\kappa > 1$ and the region containing (h, k) and bounded by the hyperbola $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$, with $a > b$, is filled with a material having refraction index n_1, and the complement of this region is filled with a material having refraction index n_2, then all rays emanating from one focus are refracted by the branch of the hyperbola closed to the other focus into rays parallel to the x-axis if the eccentricity $e = \sqrt{1+(b/a)^2} = \kappa$.

3. Existence and uniqueness of weak solutions for $\kappa < 1$

This section addresses the refractor problem in case $\kappa < 1$. We first introduce the notions of refractor mapping and measure, and weak solution. We then convert the refractor problem into an optimal mass transport problem from $\overline{\Omega}$ to $\overline{\Omega^*}$ with the cost function $\log \dfrac{1}{1-\kappa x\cdot m}$ and establish existence and uniqueness of weak solutions.

Let Ω, Ω^* be two domains on S^{n-1}, the illumination intensity of the emitting beam is given by nonnegative $f(x) \in L^1(\overline{\Omega})$, and the prescribed illumination intensity of the refracted beam is given by a nonnegative Radon measure μ on $\overline{\Omega^*}$. Throughout this section, we assume that $|\partial\Omega| = 0$ and the physical constraint

$$\inf_{x\in\Omega, m\in\Omega^*} x\cdot m \geq \kappa. \tag{3.1}$$

We further suppose that the total energy conservation

$$\int_\Omega f(x)\,dx = \mu(\overline{\Omega^*}) > 0, \tag{3.2}$$

and for any open set $G \subset \Omega$

$$\int_G f(x)\,dx > 0, \tag{3.3}$$

where dx denotes the surface measure on S^{n-1}.

3.1. **Refractor measure and weak solutions.** We begin with the notions of refractor and supporting semi-ellipsoid.

Definition 3.1. *A surface $\mathcal{R}$ parameterized by $\rho(x)x$ with $\rho \in C(\overline{\Omega})$ is a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$ for the case $\kappa < 1$ (often simply called as refractor in this section) if for any $x_0 \in \overline{\Omega}$ there exists a semi-ellipsoid $E(m,b)$ with $m \in \overline{\Omega^*}$ such that $\rho(x_0) = \dfrac{b}{1-\kappa\, m\cdot x_0}$ and $\rho(x) \leq \dfrac{b}{1-\kappa\, m\cdot x}$ for all $x \in \overline{\Omega}$. Such $E(m,b)$ is called a supporting semi-ellipsoid of $\mathcal{R}$ at the point $\rho(x_0)x_0$.*

From the definition, any refractor is globally Lipschitz on $\overline{\Omega}$.

Definition 3.2. *Given a refractor $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, the refractor mapping of $\mathcal{R}$ is the multi-valued map defined by for $x_0 \in \overline{\Omega}$*

$$\mathcal{N}_{\mathcal{R}}(x_0) = \{m \in \overline{\Omega^*} :\ E(m,b) \text{ supports } \mathcal{R} \text{ at } \rho(x_0)x_0 \text{ for some } b > 0\}.$$

Given $m_0 \in \overline{\Omega^*}$, the tracing mapping of $\mathcal{R}$ is defined by

$$\mathcal{T}_{\mathcal{R}}(m_0) = \mathcal{N}_{\mathcal{R}}^{-1}(m_0) = \{x \in \overline{\Omega} : \ m_0 \in \mathcal{N}_{\mathcal{R}}(x)\}.$$

Definition 3.3. *Given a refractor $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, the Legendre transform of $\mathcal{R}$ is defined by*

$$\mathcal{R}^* = \{\rho^*(m)m : \ \rho^*(m) = \inf_{x\in\overline{\Omega}} \frac{1}{\rho(x)(1-\kappa\, x\cdot m)},\ m \in \overline{\Omega^*}\}.$$

We now give some basic properties of Legendre transforms.

Lemma 3.4. *Let $\mathcal{R}$ be a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$. Then*

(i) *$\mathcal{R}^*$ is a refractor from $\overline{\Omega^*}$ to $\overline{\Omega}$.*
(ii) *$\mathcal{R}^{**} = (\mathcal{R}^*)^* = \mathcal{R}$.*
(iii) *If $x_0 \in \overline{\Omega}$ and $m_0 \in \overline{\Omega^*}$, then $x_0 \in \mathcal{N}_{\mathcal{R}^*}(m_0)$ iff $m_0 \in \mathcal{N}_{\mathcal{R}}(x_0)$.*

Proof. Given $m_0 \in \overline{\Omega^*}$, $\rho(x)(1-\kappa x\cdot m_0)$ must attain the maximum over $\overline{\Omega}$ at some $x_0 \in \overline{\Omega}$. Then $\rho^*(m_0) = 1/[\rho(x_0)(1-\kappa x_0\cdot m_0)]$. We always have

$$\rho^*(m) = \inf_{x\in\overline{\Omega}} \frac{1}{\rho(x)(1-\kappa m\cdot x)} \leq \frac{1}{\rho(x_0)(1-\kappa x_0\cdot m)}, \qquad \forall m \in \overline{\Omega^*}. \tag{3.4}$$

Hence $E(x_0, 1/\rho(x_0))$ is a supporting semi-ellipsoid to $\mathcal{R}^*$ at $\rho^*(m_0)m_0$. Thus, (i) is proved.

To prove (ii), from the definitions of Legendre transform and refractor mapping we have

$$\rho(x_0)\,\rho^*(m_0) = \frac{1}{1-\kappa m_0\cdot x_0} \quad \text{for } m_0 \in \mathcal{N}_{\mathcal{R}}(x_0). \tag{3.5}$$

For $x_0 \in \overline{\Omega}$, there exists $m_0 \in \mathcal{N}_{\mathcal{R}}(x_0)$ and so from (3.5) $\rho^*(m_0) = \dfrac{1/\rho(x_0)}{1-\kappa x_0\cdot m_0}$. By (3.4), $\rho^*(m)(1-kx_0\cdot m)$ attains the maximum $1/\rho(x_0)$ at m_0. Thus,

$$\rho^{**}(x_0) = \inf_{m\in\overline{\Omega^*}} \frac{1}{\rho^*(m)(1-kx_0\cdot m)} = \frac{1}{\rho(x_0)^{-1}}.$$

To prove (iii), we get from the proof of (ii) that if $m_0 \in \mathcal{N}_{\mathcal{R}}(x_0)$, then the semi-ellipsoid $E(x_0, 1/\rho(x_0))$ supports $\mathcal{R}^*$ at $\rho^*(m_0)m_0$ and so $x_0 \in \mathcal{N}_{\mathcal{R}^*}(m_0)$. On the other hand, if $x_0 \in \mathcal{N}_{\mathcal{R}^*}(m_0)$, we get that $m_0 \in \mathcal{N}_{\mathcal{R}^{**}}(x_0)$, and since $\mathcal{R}^{**} = \mathcal{R}$, $m_0 \in \mathcal{N}_{\mathcal{R}}(x_0)$. □

The next two lemmas discuss the refractor measure.

Lemma 3.5. *$\mathcal{C} = \{F \subset \overline{\Omega^*} : \mathcal{T}_{\mathcal{R}}(F)$ is Lebesgue measurable$\}$ is a σ-algebra containing all Borel sets in $\overline{\Omega^*}$.*

Proof. Obviously, $\mathcal{T}_{\mathcal{R}}(\emptyset) = \emptyset$ and $\mathcal{T}_{\mathcal{R}}(\overline{\Omega^*}) = \overline{\Omega}$. Since $\mathcal{T}_{\mathcal{R}}(\cup_{i=1}^{\infty} F_i) = \cup_{i=1}^{\infty} \mathcal{T}_{\mathcal{R}}(F_i)$, $\mathcal{C}$ is closed under countable unions. Clearly for $F \subset \overline{\Omega^*}$

$$\begin{aligned} \mathcal{T}_{\mathcal{R}}(F^c) &= \{x \in \overline{\Omega} : \mathcal{N}_{\mathcal{R}}(x) \cap F^c \neq \emptyset\} \\ &= \{x \in \overline{\Omega} : \mathcal{N}_{\mathcal{R}}(x) \cap F = \emptyset\} \cup \{x \in \overline{\Omega} : \mathcal{N}_{\mathcal{R}}(x) \cap F^c \neq \emptyset, \ \mathcal{N}_{\mathcal{R}}(x) \cap F \neq \emptyset\} \\ &= [\mathcal{T}_{\mathcal{R}}(F)]^c \cup [\mathcal{T}_{\mathcal{R}}(F^c) \cap \mathcal{T}_{\mathcal{R}}(F)]. \end{aligned} \tag{3.6}$$

If $x \in \mathcal{T}_{\mathcal{R}}(F^c) \cap \mathcal{T}_{\mathcal{R}}(F) \cap \Omega$, then $\mathcal{R}$ parameterized by ρ has two distinct supporting semi-ellipsoids $E(m_1, b_1)$ and $E(m_2, b_2)$ at $\rho(x)x$. By (2.1), $\rho(x)x$ is a singular point of $\mathcal{R}$. Otherwise, if $\mathcal{R}$ has the tangent hyperplane Π at $\rho(x)x$, then Π must coincide both with the tangent hyperplane of $E(m_1, b_1)$ and that of $E(m_2, b_2)$ at $\rho(x)x$. It follows from (2.1) that $m_1 = m_2$. Therefore, the area measure of $\mathcal{T}_{\mathcal{R}}(F^c) \cap \mathcal{T}_{\mathcal{R}}(F)$ is 0. So $\mathcal{C}$ is closed under complements, and we have proved that $\mathcal{C}$ is a σ-algebra.

To prove that $\mathcal{C}$ contains all Borel subsets, it suffices to show that $\mathcal{T}_{\mathcal{R}}(K)$ is compact if $K \subset \overline{\Omega^*}$ is compact. Let $x_i \in \mathcal{T}_{\mathcal{R}}(K)$ for $i \geq 1$. There exists $m_i \in \mathcal{N}_{\mathcal{R}}(x_i) \cap K$. Let $E(m_i, b_i)$ be the supporting semi-ellipsoid to $\mathcal{R}$ at $\rho(x_i)x_i$. We have

$$\rho(x)(1 - \kappa m_i \cdot x) \leq b_i \qquad \text{for } x \in \overline{\Omega}, \tag{3.7}$$

where the equality in (3.7) occurs at $x = x_i$. Assume that $a_1 \leq \rho(x) \leq a_2$ on $\overline{\Omega}$ for some constants $a_2 \geq a_1 > 0$. By (3.7) and (3.1), $a_1(1 - \kappa) \leq b_i \leq a_2(1 - \kappa^2)$. Assume through subsequence that $x_i \longrightarrow x_0$, $m_i \longrightarrow m_0 \in K$, $b_i \longrightarrow b_0$, as $i \longrightarrow \infty$. By taking limit in (3.7), one obtains that the semi-ellipsoid $E(m_0, b_0)$ supports $\mathcal{R}$ at $\rho(x_0)x_0$ and $x_0 \in \mathcal{T}_{\mathcal{R}}(m_0)$. This proves $\mathcal{T}_{\mathcal{R}}(K)$ is compact. □

Lemma 3.6. *Given a nonnegative $f \in L^1(\overline{\Omega})$, the set function*

$$\mathcal{M}_{\mathcal{R},f}(F) = \int_{\mathcal{T}_{\mathcal{R}}(F)} f \, dx$$

is a finite Borel measure defined on $\mathcal{C}$ and is called the refractor measure associated with $\mathcal{R}$ and f.

Proof. Let $\{F_i\}_{i=1}^{\infty}$ be a sequence of pairwise disjoint sets in $\mathcal{C}$. Let $H_1 = \mathcal{T}_{\mathcal{R}}(F_1)$, and $H_k = \mathcal{T}_{\mathcal{R}}(F_k) \setminus \cup_{i=1}^{k-1}\mathcal{T}_{\mathcal{R}}(F_i)$, for $k \geq 2$. Since $H_i \cap H_j = \emptyset$ for $i \neq j$ and $\cup_{k=1}^{\infty} H_k = \cup_{k=1}^{\infty}\mathcal{T}_{\mathcal{R}}(F_k)$, it is easy to get

$$\mathcal{M}_{\mathcal{R},f}(\cup_{k=1}^{\infty} F_k) = \int_{\cup_{k=1}^{\infty} H_k} f\,dx = \sum_{k=1}^{\infty} \int_{H_k} f\,dx.$$

Observe that $\mathcal{T}_{\mathcal{R}}(F_k) \setminus H_k = \mathcal{T}_{\mathcal{R}}(F_k) \cap (\cup_{i=1}^{k-1}\mathcal{T}_{\mathcal{R}}(F_i))$ is a subset of the singular set of $\mathcal{R}$ and has area measure 0 for $k \geq 2$. Therefore, $\int_{H_k} f\,dx = \mathcal{M}_{\mathcal{R},f}(F_k)$ and the σ-additivity of $\mathcal{M}_{\mathcal{R},f}$ follows. □

The notion of weak solutions is introduced through the conservation of energy.

Definition 3.7. *A refractor $\mathcal{R}$ is a weak solution of the refractor problem for the case $\kappa < 1$ with emitting illumination intensity $f(x)$ on $\overline{\Omega}$ and prescribed refracted illumination intensity μ on $\overline{\Omega^*}$ if for any Borel set $F \subset \overline{\Omega^*}$*

$$\mathcal{M}_{\mathcal{R},f}(F) = \int_{\mathcal{T}_{\mathcal{R}}(F)} f\,dx = \mu(F). \tag{3.8}$$

3.2. **Variational frame of optimal mass transport.** The existence and uniqueness of the refractor will be established by converting the question into a optimal mass transport problem, see [Vil03] for a comprehensive description of this field. To do so, we now turn to discuss some results for general optimal mass transport. Let D, D^* be two domains on S^{n-1} with $|\partial D| = 0$. Let $\mathcal{N}$ be a multi-valued mapping from $\overline{D}$ onto $\overline{D^*}$ such that $\mathcal{N}(x)$ is single-valued a.e. on $\overline{D}$. For $F \subset \overline{D^*}$, set $\mathcal{T}(F) = \mathcal{N}^{-1}(F) = \{x \in \overline{D} : \mathcal{N}(x) \cap F \neq \emptyset\}$. $\mathcal{N}$ is measurable if $\mathcal{T}(F)$ is Lebesgue measurable for any Borel set $F \subset \overline{D^*}$. Given nonnegative $g \in L^1(D)$ and finite Radon measure Γ on $\overline{D^*}$ satisfying $\int_D g(x)\,dx = \Gamma(\overline{D^*}) > 0$, $\mathcal{N}$ is measure preserving from $g(x)dx$ to Γ if for any Borel $F \subset \overline{D^*}$

$$\int_{\mathcal{T}(F)} g(x)\,dx = \Gamma(F). \tag{3.9}$$

Lemma 3.8. *$\mathcal{N}$ is a measure preserving mapping from $g(x)dx$ to Γ if and only if for any $v \in C(\overline{D^*})$*

$$\int_{\overline{D}} v(\mathcal{N}(x))g(x)\,dx = \int_{\overline{D^*}} v(m)\,d\Gamma(m). \tag{3.10}$$

We remark that $v(\mathcal{N}(x))$ is well defined for $x \in \overline{D} \setminus S$ where $\mathcal{N}(x)$ is single-valued on $\overline{D} \setminus S$ and $|S| = 0$, and $\int_{\overline{D}} v(\mathcal{N}(x))g(x)\,dx$ is understood as $\int_{\overline{D}\setminus S} v(\mathcal{N}(x))g(x)\,dx$.

Proof. Let $\mathcal{N}$ be a measure preserving mapping. To show (3.10), it suffices to prove it for $v = \chi_F$, the characteristic function of a Borel set F. It is easy to verify that $\chi_{\mathcal{T}(F)}(x) = \chi_F(\mathcal{N}(x))$ for $x \in \overline{D} \setminus S$. Therefore by (3.9)

$$\int_{\overline{D^*}} \chi_F(m)\,d\Gamma = \int_{\mathcal{T}(F)\cap(\overline{D}\setminus S)} g\,dx = \int_{\overline{D}\setminus S} \chi_F(\mathcal{N}(x))g(x)\,dx.$$

To prove the converse, assume that (3.10) holds. We now show for any relatively open set G in $\overline{D^*}$

$$\int_{\mathcal{T}(G)} g\,dx \le \Gamma(G). \tag{3.11}$$

Indeed, given a compact set $K \subset G$, choose $v \in C(\overline{D^*})$ such that $0 \le v \le 1$, $v = 1$ on K, and $v = 0$ outside G. By (3.10), one gets

$$\int_{\mathcal{T}(K)} g(x)\,dx \le \int_{\overline{D}} v(\mathcal{N}(x))g(x)\,dx \le \Gamma(G),$$

and (3.11) follows from arbitrariness of K. Because a Borel set can be approximated by open sets, (3.11) is still valid for Borel sets F in $\overline{D^*}$. Noticing $(\mathcal{T}(F))^c \subset \mathcal{T}(F^c)$, to get the reverse inequality we apply (3.11) to $\overline{D^*} \setminus F$ and (3.9) follows. □

Consider the general cost function $c(x,m) \in Lip(\overline{D} \times \overline{D^*})$, the space of Lipschitz functions on $\overline{D} \times \overline{D^*}$, and the set of admissible functions

$$\mathcal{K} = \{(u,v) : u \in C(\overline{D}), v \in C(\overline{D^*}), u(x) + v(m) \le c(x,m),\ \forall x \in D, \forall m \in D^*\}.$$

Define the dual functional I for $(u,v) \in C(\overline{D}) \times C(\overline{D^*})$

$$I(u,v) = \int_D u(x)g(x)\,dx + \int_{\overline{D^*}} v(m)\,d\Gamma,$$

and define the c- and c^*- transforms

$$u^c(m) = \inf_{x \in \overline{D}} [c(x,m) - u(x)], \quad m \in \overline{D^*}; \qquad v_c(x) = \inf_{m \in \overline{D^*}} [c(x,m) - v(m)], \quad x \in \overline{D}.$$

Definition 3.9. *A function $\phi \in C(\overline{D})$ is c-concave if for $x_0 \in \overline{D}$, there exist $m_0 \in \overline{D^*}$ and $b \in \mathbb{R}$ such that $\phi(x) \le c(x, m_0) - b$ on $\overline{D}$ with equality held at $x = x_0$.*

Obviously v_c is c-concave for any $v \in C(\overline{D^*})$. We collect the following properties:

(1) For any $u \in C(\overline{D})$ and $v \in C(\overline{D^*})$, $v_c \in Lip(\overline{D})$ and $u^c \in Lip(\overline{D^*})$ with Lipschitz constants bounded uniformly by the Lipschitz constant of c.

(2) If $(u, v) \in \mathcal{K}$, then $v(m) \le u^c(m)$ and $u(x) \le v_c(x)$. Also $(v_c, v), (u, u^c) \in \mathcal{K}$.

(3) ϕ is c-concave iff $\phi = (\phi^c)_c$.

Indeed, if $\phi(x) \le c(x, m_0) - b$ on $\overline{D}$ and the equality holds at $x = x_0$, then $b = \phi^c(m_0)$. So $\phi(x_0) = c(x_0, m_0) - \phi^c(m_0)$ which yields $\phi(x_0) \ge (\phi^c)_c(x_0)$. On the other hand, from the definitions of c and c^* transforms we always have that $(\phi^c)_c \ge \phi$ for any ϕ.

Definition 3.10. *Given a function $\phi(x)$, the c-normal mapping of ϕ is defined by*

$$\mathcal{N}_{c,\phi}(x) = \{m \in \overline{D^*} : \phi(x) + \phi^c(m) = c(x,m)\}, \qquad \text{for } x \in \overline{D},$$

and $\mathcal{T}_{c,\phi}(m) = \mathcal{N}^{-1}_{c,\phi}(m) = \{x \in \overline{D} : m \in \mathcal{N}_{c,\phi}(x)\}$.

We assume that the cost function $c(x, m)$ satisfies the following:

(3.12) For any c-concave function ϕ, $\mathcal{N}_{c,\phi}(x)$ is single-valued a.e. on $\overline{D}$ and $\mathcal{N}_{c,\phi}$ is Lebesgue measurable.

Lemma 3.11. *Suppose that $c(x, m)$ satisfies the assumption* (3.12). *Then*

(i) *If ϕ is c-concave and $\mathcal{N}_{c,\phi}$ is measure preserving from $g(x)dx$ to Γ, then (ϕ, ϕ^c) is a maximizer of $I(u, v)$ in $\mathcal{K}$.*

(ii) *If $\phi(x)$ is c-concave and (ϕ, ϕ^c) maximizes $I(u, v)$ in $\mathcal{K}$, then $\mathcal{N}_{c,\phi}$ is measure preserving from $g(x)dx$ to Γ.*

Proof. First prove (i). Given $(u, v) \in \mathcal{K}$, obviously

$$u(x) + v(\mathcal{N}_{c,\phi}(x)) \leq c(x, \mathcal{N}_{c,\phi}(x)) = \phi(x) + \phi^c(\mathcal{N}_{c,\phi}(x)), \qquad \text{a.e. } x \text{ on } \overline{D}.$$

Integrating the above inequality with respect to $g dx$ yields

$$\int_{\overline{D}} u g\, dx + \int_{\overline{D}} v(\mathcal{N}_{c,\phi}(x)) g(x)\, dx \leq \int_{\overline{D}} \phi g\, dx + \int_{\overline{D}} \phi^c(\mathcal{N}_{c,\phi}(x)) g(x)\, dx.$$

By Lemma 3.8, it yields $I(u, v) \leq I(\phi, \phi^c)$ and from (2) above the conclusion follows.

To prove (ii), let $\psi = \phi^c$, and for $v \in C(\overline{D^*})$, let $\psi_\theta(m) = \psi(m) + \theta\, v(m)$ where $0 < |\theta| \leq \epsilon_0$ with ϵ_0 small, and let $\phi_\theta = (\psi_\theta)_c$. It suffices to show

$$0 = \lim_{\theta \to 0} \frac{I(\phi_\theta, \psi_\theta) - I(\phi, \psi)}{\theta} = \int_{\overline{D}} -v(\mathcal{N}_{c,\phi}(x))\, g\, dx + \int_{\overline{D^*}} v(m)\, d\Gamma. \tag{3.13}$$

Since $(\phi_\theta, \psi_\theta) \in \mathcal{K}$, $I(\phi_\theta, \psi_\theta) \leq I(\phi, \psi)$. So the limit must be zero if it exists. We have

$$\frac{I(\phi_\theta, \psi_\theta) - I(\phi, \psi)}{\theta} = \int_{\overline{D}} \frac{\phi_\theta - \phi}{\theta}\, g\, dx + \int_{\overline{D^*}} v(m)\, d\Gamma.$$

To prove (3.13), one only needs to show that $\dfrac{\phi_\theta(x) - \phi(x)}{\theta}$ is uniformly bounded and $\dfrac{\phi_\theta(x) - \phi(x)}{\theta} \to -v(\mathcal{N}_{c,\phi}(x))$ for all $x \in D \setminus S$ where $\mathcal{N}_{c,\phi}(x)$ is single-valued on $D \setminus S$ and $|S| = 0$. Indeed, for $x \in D \setminus S$, we have $\phi_\theta(x) = c(x, m_\theta) - \psi_\theta(m_\theta)$ and $\phi(x) = c(x, m_1) - \psi(m_1)$ for some $m_\theta, m_1 \in \overline{D^*}$. Then we get

$$-\theta\, v(m_\theta) \leq \phi_\theta(x) - \phi(x) \leq -\theta\, v(m_1).$$

Moreover, $m_1 = \mathcal{N}_{c,\phi}(x)$ due to $\psi = \phi^c$. To finish the proof, we show that m_θ converges to m_1 as $\theta \to 0$. Otherwise, there exists a sequence m_{θ_k} such that $m_{\theta_k} \to m_\infty \neq m_1$. So $\phi(x) = c(x, m_\infty) - \psi(m_\infty)$, which yields $m_\infty \in \mathcal{N}_{c,\phi}(x)$. We then get $m_1 = m_\infty$, a contradiction. The proof is complete. □

Lemma 3.12. *There exists a c-concave ϕ such that*

$$I(\phi, \phi^c) = \sup\{I(u, v) : (u, v) \in \mathcal{K}\}.$$

Proof. Let

$$I_0 = \sup\{I(u,v) : (u,v) \in \mathcal{K}\},$$

and let $(u_k, v_k) \in \mathcal{K}$ be a sequence such that $I(u_k, v_k) \to I_0$. Set $\bar{u}_k = (v_k)_c$ and $\bar{v}_k = (\bar{u}_k)^c$. From property (2) above, $(\bar{u}_k, \bar{v}_k) \in \mathcal{K}$ and $I(\bar{u}_k, \bar{v}_k) \to I_0$. Let $c_k = \min_{\overline{D}} \bar{u}_k$ and define

$$u_k^\sharp = \bar{u}_k - c_k, \qquad v_k^\sharp = \bar{v}_k + c_k.$$

Obviously $(u_k^\sharp, v_k^\sharp) \in \mathcal{K}$ and by the mass conservation of $g dx$ and Γ, $I(\bar{u}_k, \bar{v}_k) = I(u_k^\sharp, v_k^\sharp)$. Since $\bar{u}_k$ are uniformly Lipschitz, $u_k^\sharp$ are uniformly bounded. In addition, $v_k^\sharp = (\bar{u}_k)^c + c_k = (u_k^\sharp)^c$ and consequently $v_k^\sharp$ are also uniformly bounded. By Arzelá-Ascoli's theorem, $(u_k^\sharp, v_k^\sharp)$ contains a subsequence converging uniformly to (ϕ, ψ) on $\overline{D} \times \overline{D^*}$. We then obtain that $(\phi, \psi) \in \mathcal{K}$ and $I_0 = \sup\{I(u,v) : (u,v) \in \mathcal{K}\} = I(\phi, \psi)$. From property (2) above, $(\psi_c, (\psi_c)^c)$ is the sought maximizer of $I(u,v)$. □

Lemma 3.13. *Suppose that $c(x,m)$ satisfies the assumption* (3.12). *Let (ϕ, ϕ^c) with ϕ c-concave be a maximizer of $I(u,v)$ in $\mathcal{K}$. Then $\inf_{s \in \mathcal{S}} \int_{\overline{D}} c(x, s(x)) g(x)\, dx$ is attained at $s = \mathcal{N}_{c,\phi}$, where $\mathcal{S}$ is the class of measure preserving mappings from $g(x)dx$ to Γ. Moreover*

$$\inf_{s \in \mathcal{S}} \int_{\overline{D}} c(x, s(x)) g(x)\, dx = \sup\{I(u,v) : (u,v) \in \mathcal{K}\}. \tag{3.14}$$

Proof. Let $\psi = \phi^c$. For $s \in \mathcal{S}$, we have

$$\begin{aligned}
\int_{\overline{D}} c(x, s(x)) g(x)\, dx &\geq \int_{\overline{D}} \Big(\phi(x) + \psi(s(x))\Big)\, g(x)\, dx \\
&= \int_{\overline{D}} \phi(x) g(x)\, dx + \int_{\overline{D}} \psi(s(x)) g(x)\, dx \\
&= \int_{\overline{D}} \phi(x) g(x)\, dx + \int_{\overline{D^*}} \psi(m)\, d\Gamma = I(\phi, \psi) \\
&= \int_{\overline{D}} \Big(\phi(x) + \psi(\mathcal{N}_{c,\phi}(x))\Big)\, g(x)\, dx, \text{ from Lemma 3.11(ii)} \\
&= \int_{\overline{D}} c(x, \mathcal{N}_{c,\phi}(x)) g(x)\, dx.
\end{aligned}$$

□

Obviously, for any c-concave function ϕ, $\mathcal{N}_{c,\phi}$ has the following converging property (C): if $m_k \in \mathcal{N}_{c,\phi}(x_k)$, $x_k \longrightarrow x_0$ and $m_k \longrightarrow m_0$, then $m_0 \in \mathcal{N}_{c,\phi}(x_0)$.

Lemma 3.14. *Assume that $c(x,m)$ satisfies the assumption* (3.12) *and that $\int_G g\,dx > 0$ for any open $G \subset D$. Then the minimizing mapping of $\inf_{s\in\mathcal{S}} \int_{\overline{D}} c(x, s(x))g(x)\,dx$ is unique in the class of measure preserving mappings from $g(x)dx$ to Γ with the converging property (C).*

Proof. From Lemmas 3.12 and 3.13, let $\mathcal{N}_{c,\phi}$ be a minimizing mapping associated with a maximizer (ϕ, ϕ^c) of $I(u, v)$ with ϕ c-concave. Suppose that $\mathcal{N}_0$ is another minimizing mapping with the converging property (C). Clearly

$$\begin{aligned}
&\int_{\overline{D}} \Big(c(x, \mathcal{N}_0(x)) - \phi(x) - \phi^c(\mathcal{N}_0(x))\Big)\, g(x)\,dx \\
&= \inf_{s\in\mathcal{S}} \int_{\overline{D}} c(x, s(x))g(x)\,dx - \left(\int_{\overline{D}} \phi(x)g(x)\,dx + \int_{\overline{D^*}} \phi^c(m)\,d\Gamma\right) = 0,
\end{aligned}$$

and since $\phi(x)+\phi^c(\mathcal{N}_0(x)) \le c(x, \mathcal{N}_0(x))$, it follows that $\phi(x)+\phi^c(\mathcal{N}_0(x)) = c(x, \mathcal{N}_0(x))$ on the set $\{x \in D : g(x) > 0\}$ which is dense in D. Hence from (3.12) and the converging property (C), we get $\mathcal{N}_0(x) = \mathcal{N}_{c,\phi}(x)$ a.e. on D. □

We remark from the above proof that if $g(x) > 0$ on D, then the minimizing mapping of $\inf_{s\in\mathcal{S}} \int_{\overline{D}} c(x, s(x))g(x)\,dx$ is unique in the class of measure preserving mappings from $g(x)dx$ to Γ.

3.3. **Existence and uniqueness for the refractor problem if $\kappa < 1$.** We are ready to use the concepts and results discussed above to establish the following main existence and uniqueness theorem.

Theorem 3.15. *Assume that Ω, Ω^*, f, μ satisfy* (3.1)-(3.3) *and $|\partial\Omega| = 0$. Then there exists a weak solution $\mathcal{R}$ unique up to dilations of the refractor problem for the case $\kappa < 1$ with emitting illumination intensity $f(x)$ and prescribed refracted illumination intensity μ.*

Proof. We first transform our problem into an optimal mass transport problem for the cost function $c(x, m) = \log \dfrac{1}{1-\kappa x\cdot m}$. Obviously, $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$ is a refractor iff $\log\rho$ is c-concave. Furthermore, $\log\rho^* = (\log\rho)^c$, $\log\rho = (\log\rho^*)_c$ by Remark (3) after Definition 3.9, and $\mathcal{N}_{\mathcal{R}}(x_0) = \mathcal{N}_{c,\log\rho}(x_0)$ by (3.5). By the Snell law and Lemma 3.5, $c(x, m)$ satisfies (3.12). From the definitions, $\mathcal{R}$ is a weak solution of the refractor problem iff $\log\rho$ is c-concave and $\mathcal{N}_{c,\log\rho}$ is a measure preserving mapping from $f(x)dx$ to μ.

By Lemma 3.12, there exists a c-concave $\phi(x)$ such that (ϕ, ϕ^c) maximizes

$$I(u, v) = \int_{\overline{\Omega}} u f\, dx + \int_{\overline{\Omega^*}} v\, d\mu(m)$$

in $\mathcal{K} = \{(u, v) \in C(\overline{\Omega}) \times C(\overline{\Omega^*}) : u(x) + v(m) \le c(x, m),\ \text{for } x \in \overline{\Omega}, m \in \overline{\Omega^*}\}$. Then by Lemma 3.11, $\mathcal{N}_{c,\phi}(x)$ is a measure preserving mapping from $f dx$ to μ. Therefore, $\mathcal{R} = \{e^{\phi(x)}x : x \in \overline{\Omega}\}$ is a weak solution of the refractor problem.

It remains to prove the uniqueness of solutions up to dilations. Let $\mathcal{R}_i = \{\rho_i(x)x : x \in \overline{\Omega}\}$, $i = 1, 2$, be two weak solutions of the refractor problem. Obviously, $\mathcal{N}_{c,\log\rho_i}$ have the converging property (C). It follows from Lemmas 3.11, 3.13 and 3.14 that $\mathcal{N}_{c,\log\rho_1}(x) = \mathcal{N}_{c,\log\rho_2}(x)$ a.e. on Ω. That is, $\mathcal{N}_{\mathcal{R}_1}(x) = \mathcal{N}_{\mathcal{R}_2}(x)$ a.e. on Ω. By (2.1), $\nu_i(x) = \dfrac{x - \kappa\mathcal{N}_{\mathcal{R}_i}(x)}{|x - \kappa\mathcal{N}_{\mathcal{R}_i}(x)|}$ is the unit normal to $\mathcal{R}_i$ towards medium II at $\rho_i(x)x$ where $\mathcal{R}_i$ is differentiable. So $\nu_1(x) = \nu_2(x)$ a.e. and consequently $\rho_1(x) = C\,\rho_2(x)$ for some $C > 0$. □

4. Existence and uniqueness of weak solutions in case $\kappa > 1$

The refractor problem in case $\kappa > 1$ can be solved by converting it to an optimal mass transport problem in a way similar to the case $\kappa < 1$. The main difference is to use semi-hyperboloids of two sheets $H(m, b)$, defined by (2.9), in place of the semi-ellipsoids $E(m, b)$, and the associated cost function for the case $\kappa > 1$ is given by $\log(\kappa x \cdot m - 1)$.

Let Ω, Ω^* be two domains on S^{n-1}, the illumination intensity of the emitting beam is given by nonnegative $f(x) \in L^1(\overline{\Omega})$, and the prescribed illumination intensity of the refracted beam is given by a nonnegative Radon measure μ on $\overline{\Omega^*}$. Throughout this section, we assume that $|\partial\Omega| = 0$ and the physical constraint

$$\inf_{x\in\Omega, m\in\Omega^*} x \cdot m \geq 1/\kappa + \delta, \tag{4.1}$$

for some $\delta > 0$. We also suppose that the total energy conservation

$$\int_\Omega f(x)\, dx = \mu(\overline{\Omega^*}) > 0, \tag{4.2}$$

and for any open set $G \subset \Omega$

$$\int_G f(x)\, dx > 0. \tag{4.3}$$

We begin with the notions of refractor and supporting semi-hyperboloid.

Definition 4.1. *A surface $\mathcal{R}$ parameterized by $\rho(x)x$ with $\rho \in C(\overline{\Omega})$ is a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$ for the case $\kappa > 1$ (often simply called as refractor in this section) if for any $x_0 \in \overline{\Omega}$ there exists a semi-hyperboloid $H(m, b)$ with $m \in \overline{\Omega^*}$ such that $\rho(x_0) = \dfrac{b}{\kappa\, m \cdot x_0 - 1}$ and $\rho(x) \geq \dfrac{b}{\kappa\, m \cdot x - 1}$ for all $x \in \overline{\Omega}$. Such $H(m, b)$ is called a supporting semi-hyperboloid of $\mathcal{R}$ at the point $\rho(x_0)x_0$.*

Obviously, any refractor must be Lipschitz in $\overline{\Omega}$. The refractor mapping and Legendre transform are defined similarly.

Definition 4.2. *Given a refractor $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, the refractor mapping of $\mathcal{R}$ is the multi-valued map defined by for $x_0 \in \overline{\Omega}$*

$$\mathcal{N}_{\mathcal{R}}(x_0) = \{m \in \overline{\Omega^*} :\ H(m, b) \text{ supports } \mathcal{R} \text{ at } \rho(x_0)x_0 \text{ for some } b > 0\}.$$

Given $m_0 \in \overline{\Omega^*}$, the tracing mapping of $\mathcal{R}$ is defined by

$$\mathcal{T}_{\mathcal{R}}(m_0) = \mathcal{N}_{\mathcal{R}}^{-1}(m_0) = \{x \in \overline{\Omega} :\ m_0 \in \mathcal{N}_{\mathcal{R}}(x)\}.$$

Definition 4.3. *Given a refractor $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, the Legendre transform of $\mathcal{R}$ is defined by*

$$\mathcal{R}^* = \{\rho^*(m)m :\ \rho^*(m) = \sup_{x\in\overline{\Omega}} \frac{1}{\rho(x)(\kappa\, x \cdot m - 1)},\ m \in \overline{\Omega^*}\}.$$

The proof of the following lemma is analogous to that of Lemma 3.4.

Lemma 4.4. *Let $\mathcal{R}$ be a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$. Then*

(i) *$\mathcal{R}^*$ is a refractor from $\overline{\Omega^*}$ to $\overline{\Omega}$.*

(ii) *$\mathcal{R}^{**} = (\mathcal{R}^*)^* = \mathcal{R}$.*

(iii) *If $x_0 \in \overline{\Omega}$ and $m_0 \in \overline{\Omega^*}$, then $x_0 \in \mathcal{N}_{\mathcal{R}^*}(m_0)$ iff $m_0 \in \mathcal{N}_{\mathcal{R}}(x_0)$.*

We can also prove the following lemma about the refractor measure.

Lemma 4.5. *Let $\mathcal{R}$ be a refractor. Then $C = \{F \subset \overline{\Omega^*} : \mathcal{T}_{\mathcal{R}}(F)$ is Lebesgue measurable$\}$ is a σ-algebra containing all Borel sets in $\overline{\Omega^*}$, and the set function on C defined by*

$$\mathcal{M}_{\mathcal{R},f}(F) = \int_{\mathcal{T}_{\mathcal{R}}(F)} f\,dx$$

is a finite Radon measure, called as the refractor measure associated with $\mathcal{R}$ and f.

Weak solutions are again introduced through energy conservation.

Definition 4.6. *A refractor $\mathcal{R}$ is a weak solution of the refractor problem for the case $\kappa > 1$ with emitting illumination intensity $f(x)$ on $\overline{\Omega}$ and prescribed refracted illumination intensity μ on $\overline{\Omega^*}$ if for any Borel set $F \subset \overline{\Omega^*}$*

$$\mathcal{M}_{\mathcal{R},f}(F) = \int_{\mathcal{T}_{\mathcal{R}}(F)} f\,dx = \mu(F). \tag{4.4}$$

Below is the main existence and uniqueness theorem for the case $\kappa > 1$.

Theorem 4.7. *Assume that Ω, Ω^*, f, μ satisfy* (4.1)-(4.3) *and $|\partial\Omega| = 0$. Then there exists a weak solution $\mathcal{R}$ unique up to dilations of the refractor problem for the case $\kappa > 1$ with emitting illumination intensity $f(x)$ and prescribed refracted illumination intensity μ.*

Proof. We first convert the problem into an optimal mass transport problem with the cost function $c(x, m) = \log(\kappa x \cdot m - 1)$. Obviously, $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$ is a refractor iff $-\log \rho$ is c-concave. Furthermore, $-\log \rho^* = (-\log \rho)^c$, $-\log \rho = (-\log \rho^*)_c$ by Lemma 4.4, and $\mathcal{N}_{\mathcal{R}}(x_0) = \mathcal{N}_{c,-\log \rho}(x_0)$. By the Snell law, $c(x, m)$

satisfies (3.12). From the definitions, $\mathcal{R}$ is a weak solution of the refractor problem iff $-\log\rho$ is c-concave and $\mathcal{N}_{c,-\log\rho}$ is a measure preserving mapping from $f dx$ to μ.

By Lemma 3.12, there exists a c-concave function ϕ such that (ϕ, ϕ^c) maximizes

$$I(u, v) = \int_{\overline{\Omega}} u f \, dx + \int_{\overline{\Omega^*}} v \, d\mu(m)$$

in $\mathcal{K} = \{(u, v) \in C(\overline{\Omega}) \times C(\overline{\Omega^*}) : u(x) + v(m) \leq c(x, m), \text{ for } x \in \overline{\Omega}, m \in \overline{\Omega^*}\}$. Then by Lemma 3.11, $\mathcal{N}_{c,\phi}(x)$ is a measure preserving mapping from $f dx$ to μ. Therefore, $\mathcal{R} = \{e^{-\phi(x)}x : x \in \overline{\Omega}\}$ is a weak solution of the refractor problem.

It remains to prove the uniqueness of solutions up to dilations. Let $\mathcal{R}_i = \{\rho_i(x)x : x \in \overline{\Omega}\}, i = 1, 2$, be two weak solutions of the refractor problem. Obviously, $\mathcal{N}_{c,-\log\rho_i}$ has the converging property (C). It follows from Lemmas 3.11, 3.13 and 3.14 that $\mathcal{N}_{c,-\log\rho_1}(x) = \mathcal{N}_{c,-\log\rho_2}(x)$ a.e. on Ω. That is, $\mathcal{N}_{\mathcal{R}_1}(x) = \mathcal{N}_{\mathcal{R}_2}(x)$ a.e. on Ω. By (2.1), $\nu_i(x) = \dfrac{\kappa\mathcal{N}_{\mathcal{R}_i}(x) - x}{|\kappa\mathcal{N}_{\mathcal{R}_i}(x) - x|}$ is the unit normal to $\mathcal{R}_i$ towards medium II at $\rho_i(x)x$ where $\mathcal{R}_i$ is differentiable. So $\nu_1(x) = \nu_2(x)$ a.e. and consequently $\rho_1(x) = C\rho_2(x)$ for some $C > 0$. □

Remark 4.8. Theorems 3.15 and 4.7 imply the existence of a lens refracting radiation in a prescribed way. The lens is bounded by two surfaces, the "outer" surface is the one described in those theorems and the "inner" one is a sphere with center at the point from where the radiation emanates. These ideal lenses do not have spherical aberration, i.e., they focus all incoming rays into exactly one point.

5. The differential equation and condition A3

If f is a function defined in an n-dimensional neighborhood of the point $x \in S^{n-1}$, the tangential gradient f at x is defined by

$$\nabla_x f(x) = D_x f(x) - (D_x f(x) \cdot x)\, x,$$

where D_x is the standard gradient in $\mathbb{R}^n$. Also the tangential Hessian of f at x with respect to the standard metric in the sphere, $dx_1^2 + \cdots + dx_n^2$, is defined by

$$\nabla^2_{xx} f(x) = D^2_{xx} f(x) - (D_x f(x) \cdot x) Id,$$

where D^2_{xx} is the standard Hessian in $\mathbb{R}^n$.

Since the refractor problem is a mass transportation problem, the differential equation satisfied by the potential $\phi = \log \rho$ is

$$\det\left(\nabla^2_{xx}\phi(x) + \mathcal{A}(x,p)\right) = \mathcal{B}(x,p), \quad p = \nabla_x \phi(x), \tag{5.1}$$

with $\mathcal{A}(x,p) = \left(\nabla^2_{x_k x_l} c(x, Y(x,p))\right)$, $\mathcal{B}(x,p) = \left|\det\left(\nabla_{x_k}\nabla_{y_l} c(x, Y(x,p))\right)\right| \dfrac{f(x)}{f^*(Y(x,p))}$, and $\nabla_x c(x,Y) = p$; see [MTW05], [TW07], [Tru06], and [Vil07, Chapter 12]. We are going to calculate the matrix $\mathcal{A}$ and $\mathcal{B}$.

Let $0 < \kappa < 1$, and the cost function $c(x,y) = -\frac{1}{\kappa}\log(1 - \kappa\, x \cdot y)$, $x \cdot y \geq \kappa$ (the factor $1/\kappa$ is added to simplify the computations). We have

$$\nabla_x c(x,y) = \frac{y - (x \cdot y)\, x}{1 - \kappa\, x \cdot y}.$$

For each $x \in S^{n-1}$ and for each $p \in \mathbb{R}^n$ in the tangent plane to the sphere at x, i.e., $x \perp p$, we want to find $Y = Y(x,p) \in S^{n-1}$ such that $\nabla_x c(x,Y) = p$. Indeed, we write $Y = \lambda x + \mu p + \delta x^\perp$ with $x^\perp$ a vector orthogonal to both x and p. Then $\nabla_x c(x,Y) = \dfrac{\mu\, p + \delta x^\perp}{1 - \kappa\lambda} = p$. We have $\delta = 0$, $1 = \lambda^2 + \mu^2|p|^2$, and since $x \cdot Y \geq \kappa$, we obtain

$$Y(x,p) = \lambda(p)\, x + (1 - \kappa\, \lambda(p))\, p, \qquad \lambda(p) = \frac{\kappa|p|^2 + \sqrt{h(p)}}{1 + \kappa^2|p|^2}, \qquad h(p) = 1 - (1-\kappa^2)|p|^2, \tag{5.2}$$

for $|p| \leq 1/\sqrt{1-\kappa^2}$ (there is no solution Y for larger values of p). Notice that $D_x c(x,Y) \cdot x = \dfrac{\lambda(p)}{1 - \kappa\, \lambda(p)}$ and

$$c_{x_i}(x,Y) = p_i + \frac{\lambda(p)}{1 - \kappa\, \lambda(p)} x_i. \tag{5.3}$$

We next calculate $\nabla^2_{xx}c(x,y) = \left(\nabla^2_{x_kx_l}c(x,y)\right)$. Notice that

$$c_{x_kx_l}(x,y) = \kappa c_{x_k}(x,y)\, c_{x_l}(x,y).$$

We have $\mathcal{A}(x,p) = (a_{kl}(x,p)) = \left(\nabla^2_{x_kx_l}c(x,Y(x,p))\right)$ and therefore

$$a_{kl}(x,p) = \kappa\left(p_k + \frac{\lambda(p)}{1-\kappa\,\lambda(p)}x_k\right)\left(p_l + \frac{\lambda(p)}{1-\kappa\,\lambda(p)}x_l\right) - \frac{\lambda(p)}{1-\kappa\,\lambda(p)}\delta_{kl}. \tag{5.4}$$

The A3s condition introduced by Ma, Trudinger and Wang [MTW05] reads

$$D_{p_ip_j}a_{kl}(x,p)\xi_i\xi_j\eta_k\eta_l \le -c_0|\xi|^2|\eta|^2, \tag{5.5}$$

for all $x \in S^{n-1}$, and for all $p,\xi,\eta \in \mathbb{R}^n$ in the tangent plane to x, i.e., $p,\xi,\eta \perp x$, with $\xi \perp \eta$, and c_0 a positive constant. Similarly, the matrix $\mathcal{A}$ verifies A3w if (5.5) holds with $c_0 = 0$. Letting $g(p) = \dfrac{\lambda(p)}{1-\kappa\,\lambda(p)} = \dfrac{\kappa}{1-\kappa^2} + \dfrac{\sqrt{h(p)}}{1-\kappa^2}$ and differentiating a_{kl} with respect to p_i, p_j yields

$$\begin{aligned} D_{p_ip_j}a_{kl}(x,p) = \kappa\Big((\partial_{p_ip_j}g\, x_k)(p_l + gx_l) + (\delta_{ki} + \partial_{p_i}gx_k)(\delta_{lj} + \partial_{p_j}gx_l) \\ +(\delta_{kj} + \partial_{p_j}gx_k)(\delta_{li} + \partial_{p_i}gx_l) + (p_k + gx_k)(\partial_{p_ip_j}gx_l)\Big) - \partial_{p_ip_j}g\delta_{kl}. \end{aligned}$$

Hence

$$\begin{aligned} &D_{p_ip_j}a_{kl}(x,p)\xi_i\xi_j\eta_k\eta_l \\ &= 2\kappa\langle D^2_{pp}g(p)\xi,\xi\rangle(x\cdot\eta)\left((p\cdot\eta) + g(p)(x\cdot\eta)\right) + 2\kappa(x\cdot\eta)^2\langle D_pg(p)\cdot\xi\rangle^2 - \langle D^2_{pp}g(p)\xi,\xi\rangle|\eta|^2. \end{aligned}$$

We have

$$\langle D^2_{pp}g(p)\xi,\xi\rangle = -\frac{|\xi|^2h(p) + (1-\kappa^2)(p\cdot\xi)^2}{h(p)^{3/2}}$$

Since $x \perp \xi,\eta$, and $\xi \perp \eta$, we obtain

$$\sum_{i,j,k,l} D_{p_ip_j}a_{kl}(x,p)\xi_i\xi_j\eta_k\eta_l = \frac{|\xi|^2h(p) + (1-\kappa^2)(p\cdot\xi)^2}{h(p)^{3/2}}\,|\eta|^2,$$

and so A3w does not hold[‡]. From [Loe06, Theorem 3.2] one cannot expect C^1 regularity of u in case $\kappa < 1$.

[‡]Notice that $-c$ verifies A3w for $|p| \le 1/\sqrt{1-\kappa^2}$.

Let us now consider the case $\kappa > 1$. In this case the cost function is $c(x, y) = \dfrac{1}{\kappa}\log(\kappa\, x \cdot y - 1)$, for $x \cdot y \geq 1/\kappa$. We then have

$$\nabla_x c(x, y) = \frac{y - (x \cdot y)x}{\kappa x \cdot y - 1}.$$

Proceeding as in the previous case we get that

$$Y(x, p) = \lambda(p)x + (\kappa\lambda(p) - 1)p,$$

where $\lambda(p)$ is defined in (5.2), now for all p. Since in this case

$$c_{x_i x_j}(x, y) = -\kappa c_{x_i}(x, y)\, c_{x_j}(x, y),$$

the formula for $\mathcal{A}(x, p)$ is now

$$a_{kl}(x, p) = -\kappa\left(p_k + \frac{\lambda(p)}{\kappa\,\lambda(p) - 1}x_k\right)\left(p_l + \frac{\lambda(p)}{\kappa\,\lambda(p) - 1}x_l\right) - \frac{\lambda(p)}{\kappa\,\lambda(p) - 1}\delta_{kl},$$

that is,

$$a_{kl}(x, p) = -\kappa\,(p_k - g(p)x_k)\,(p_l - g(p)x_l) + g(p)\delta_{kl}.$$

Therefore we now get

$$\begin{aligned}&D_{p_i p_j} a_{kl}(x, p)\xi_i\xi_j\eta_k\eta_l\\ &= -2\kappa\langle D^2_{pp}g(p)\xi, \xi\rangle(x \cdot \eta)\,((p \cdot \eta) - g(p)(x \cdot \eta)) - 2\kappa(x \cdot \eta)^2\langle D_p g(p) \cdot \xi\rangle^2 + \langle D^2_{pp}g(p)\xi, \xi\rangle|\eta|^2.\end{aligned}$$

From the orthogonality and the value of $D^2_{pp}g$,

$$\sum_{i,j,k,l} D_{p_i p_j} a_{kl}(x, p)\xi_i\xi_j\eta_k\eta_l = -\frac{|\xi|^2 h(p) + (1 - \kappa^2)(p \cdot \xi)^2}{h(p)^{3/2}}\,|\eta|^2 \leq -\frac{1}{h(p)^{3/2}}|\xi|^2|\eta|^2,$$

and therefore A3w holds and A3s does not.

To find $\mathcal{B}$, if $0 < \kappa < 1$, then we have

$$\nabla_{x_i}\nabla_{y_j} c(x, y) = \frac{\delta_{ij}}{1 - \kappa x \cdot y} + \kappa\frac{y_i x_j}{(1 - \kappa x \cdot y)^2} - \frac{x_i x_j}{(1 - \kappa x \cdot y)^2} - \frac{y_i y_j}{1 - \kappa x \cdot y} - \frac{\kappa y_i y_j (x \cdot y)}{(1 - \kappa x \cdot y)^2} + \frac{x_i y_j (x \cdot y)}{(1 - \kappa x \cdot y)^2}.$$

Therefore

$$\begin{aligned}\nabla_{x_i}\nabla_{y_j}c(x,Y(x,p)) &= \frac{\delta_{ij}}{1-\kappa\lambda} + \frac{x_j(\kappa Y_i - x_i) - Y_j(Y_i - \lambda x_i)}{(1-\kappa\lambda)^2} \\ &= \frac{\delta_{ij}}{1-\kappa\lambda} + \frac{1}{1-\kappa\lambda} x_j\left((\kappa-\lambda)p_i - x_i\right) - p_i p_j.\end{aligned}$$

Similarly, if $\kappa > 1$ one obtains that

$$\nabla_{x_i}\nabla_{y_j}c(x,Y(x,p)) = \frac{\delta_{ij}}{\kappa\lambda-1} + \frac{1}{\kappa\lambda-1} x_j\left((\lambda-\kappa)p_i - x_i\right) + p_i p_j.$$

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