REFLECTOR PROBLEM IN $\mathbb{R}^n$ ENDOWED WITH NON-EUCLIDEAN NORM

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Abstract. In this paper, we establish existence and uniqueness up to dilations for the reflector problem in a nonisotropic medium in $\mathbb{R}^n$ for which light wavefronts are described by non-Euclidean norm spheres, through approaches of paraboloid approximation and optimal mass transport.

§1. Introduction

Recently, there have been a lot of interests in the mathematical study of the reflector problem originated in the synthesis of reflector antennas in engineering. Let O be a nonisotropic point source of light in a homogeneous isotropic medium. By the law of reflection, if a light ray emanating from O in the direction $x \in S^{n-1}$ is incident upon a perfectly reflecting surface $\mathcal{R}$, then the reflected ray has direction:

$$m = \mathcal{N}(x) = x - 2(x\nu)\nu, \tag{1-1}$$

where ν is the unit outer normal to $\mathcal{R}$ at the point where the light ray hits $\mathcal{R}$. Throughout this paper, $x\nu$ donotes the inner product of x and ν in $\mathbb{R}^n$. The reflector problem can be described in the following. Given Ω, $\Omega^* \subset S^{n-1}$, and two positive functions $f(x)$, $g(m)$ defined on Ω, Ω^*, respectively, we want to construct a reflecting surface $\mathcal{R}$ parameterized by $X = \rho(x)x$ for $x \in \Omega$ such that lights emitted from O with aperture Ω and input illumination intensity $f(x)$ are reflected by $\mathcal{R}$ into the directions in Ω^* and all reflected lights cover Ω^* with prescribed output illumination intensity $g(m)$. By conservation of energy, $\mathcal{N}$ in (1-1) is measure-preserving:

$$\int_{\mathcal{N}^{-1}(E)} f(x)\,dx = \int_E g(m)\,dm, \qquad \text{for all Borel set } E \subset \Omega^*.$$

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By [GW98], it yields the following fully nonlinear equation on S^{n-1}

$$\frac{\det\left(\nabla_{ij}u+(u-\eta)e_{ij}\right)}{\eta^{n-1}\det(e_{ij})}=\frac{f(x)}{g(\mathcal{N}(x))}, \tag{1-2}$$

where $u=1/\rho$, ∇=covariant derivative, $\eta=\dfrac{|\nabla u|^2+u^2}{2u}$, and e is the metric on S^{n-1}. An equation similar to (1-2) for the Legendre transforms was derived earlier when $n=3$. See [OW87].

The reflector problem in the case that $n=3$, and Ω and Ω^* are contained in the northern and southern hemispheres respectively, was discussed in [Wan96], [Wan04]. The existence and uniqueness up to dilations of weak solutions were established in [Wan96]. Regularity of weak solutions was also discussed in [Wan96] if f, g are smooth and Ω, Ω^* satisfy certain geometric condition. A Monge-Kantorovich optimal transport approach was found and used to study this reflector problem in [Wan04].

For the global reflector problem (i.e., $\Omega=\Omega^*=S^{n-1}$), the existence of weak solutions was established in [CO94] if f and g are nonnegative L^1 functions, and the uniqueness up to homothety was treated in [GW98] under additional assumption that $\int_G g(m)\,dm>0$ for any open set G in S^{n-1}. The existence of the optimal (reflector) map was discussed in [GaO07]. If f, g are strictly positive and in $C^{1,1}(S^{n-1})$, then weak solutions are $C^{3,\alpha}$ for any $0<\alpha<1$, by [GW98]. Moreover, there has been some important progress in the study of properties of weak solutions to the global reflector problem. In [CGH05], the inequalities of Alexandrov type for the reflector map were established, and then they were used to prove that any weak solution is a strict reflector (i.e., each supporting paraboloid touches it at only one point) and is of C^1 class, under the assumption that f and g are bounded away from 0 and ∞. In [O05], the focal function of reflectors was discussed and characterized.

In this paper, we consider the global reflector problem in a homogeneous nonisotropic medium. This is, for instance, the case when the effective speed of propagation in a periodic medium is considered. Indeed, given a periodic medium (for instance a crystal) of period ε, the geodesic distance between two points x and y is translation invariant up to an error of order ε, since given a vector z, $x+z$ and $y+z$ differ from x and y in vector of order ε modulo the lattice ($z=\bar{z}+\varepsilon v$ for $\bar{z}$ a vector of the lattice). Therefore, as ε goes to zero, the geodesic distance becomes translation invariant and the distance to the origin a norm. On the other hand, every norm can be realized as the homogenization limit of a periodic medium. See [AB84], [BBF02].

Suppose that wave fronts of lights transmitted from a point source in this medium are a family of closed surfaces homothetic to a C^1 closed strictly convex surface Σ symmetric with respect to O, and radiation (light ray) flows along straight lines. The optical distance for radiation to flow from O to X, i.e., time taken up to a scale, can be represented by the norm $\|X\|$ induced by Σ (Minkowski functional of Σ). A radiation emitted from O incident upon X_0 on a reflecting hyperplane Π is reflected according to Fermat's principle. Thus, if the reflected radiation passes through Y_0, then the optical path OX_0Y_0 has the least optical length among all paths OXY_0 for $X\in\Pi$, i.e., $\|X_0\|+\|Y_0-X_0\|=\inf_{X\in\Pi}(\|X\|+\|Y_0-X\|)$. In general, the reflection angle is not necessarily equal to the incident angle.

In fact, the natural relation between the variables x and m below is not that of variables in the norm space B but in dual spaces B and B^*. While the geometry associated to x is that of the norm $\|\cdot\|$ given by effective distance, that associated to m is the one of the dual norm $\|\cdot\|_*$. It is like optimal mass transport for nonisotropic propagation speeds goes from B to B^*, by identifying x with m through the support function (the Hahn-Banach theorem that assigns to the unit vector x, the unit form m_x that realizes its norm $x \cdot m_x = 1$). This suggests considering the infinitely dimensional problem. We will come back to this issue in a later paper.

The global reflector problem in this medium can be described below. Let $\|\cdot\|$ be the norm in $\mathbb{R}^n$ such that the unit norm sphere Σ is C^1 strictly convex and norm spheres represent wavefronts. Given two nonnegative functions $f(x)$, $g(m)$ defined on Σ, we want to construct a reflecting surface $\mathcal{R}$ parameterized by $X = \rho(x)x$ for $x \in \Sigma$ such that lights emitted from O with input illumination intensity $f(x)$ are reflected by $\mathcal{R}$ according to Fermat's principle and the reflected lights have prescribed output illumination intensity $g(m)$.

Our purpose in this paper is to deal with existence and uniqueness of this global generalized reflector problem (called as Σ-reflector problem). In §3-§6, we develop a very general theory via paraboloid approximation and geometric argument if Σ is only C^1 strictly convex. In §7, we use the optimal mass transport method to reconsider this global Σ-reflector problem if Σ is C^2 strongly convex.

Our work in §3-§5 is inspired by [CO94]. The development of the notion of weak solutions and existence theory is parallel to that in [CO94]. However, the Σ-reflector problem is much more involved with convex analysis. We formulate the problem by using convex analysis and the Fermat principle. This formulation is more involved technically than that in isotropic media and part of it is new.

For uniqueness, we extend the idea in [GW98, Theorem 3.2] to the Σ-reflector problem. The argument in [GW98] relies on some geometrical intuition with few details, and their notion of weak solutions is generally not equivalent to ours when $\Sigma = S^{n-1}$. The generality of the Σ-reflector problem requires a more analytically detailed argument presented in §6.

The optimal mass transport method was first used by [Wan04] for the local standard reflector problem ($\Sigma = S^{n-1}$ and $\overline{\Omega} \cap \overline{\Omega}^* = \emptyset$) for which the cost function $-\log(1 - xm) = -\log \frac{|x-m|^2}{2}$ is smooth. Also see [GlO03]. However, for the global case, this cost function is singular, and the existence under the framework of optimal mass transport is more difficult. Recently, [GaO07] considered the optimal mass transport problem from S^{n-1} to S^{n-1} for more general singular cost functions $c(x, m) = l(\frac{|x-m|^2}{2})$ with $l(0^+) = \infty$ and proved the existence of optimal maps by using c-cyclical monotonicity. In §7, we provide variational arguments for the global Σ-reflector problem which has a singular cost function $c(x, m^*) = -\log(1 - xm^*)$ defined on $\Sigma \times \Sigma^*$, where Σ^* is the unit dual norm sphere. Our approaches are different from that in [GaO07] and through the dual functional, the first way by maximizing sequence and second way by cost function approximation.

§2. Preliminary on Convexity

In this section, we give preliminary on some notions and facts about convexity. For the convenience of readers, the proofs for most of these facts are presented.

Given a norm $\|\cdot\|$ in $\mathbb{R}^n$ $(n \geq 2)$, let $\Sigma = \{x : \|x\| = 1\}$ be the unit norm sphere and let $\mathbf{B} = \{x : \|x\| \leq 1\}$ be the unit norm ball. Throughout the paper, denote by $|x|$ the standard Euclidean norm of x and by $S^{n-1} = \{x : |x| = 1\}$ the Euclidean unit sphere. Obviously, Σ is a closed convex surface enclosing the origin O and symmetric with respect to O. If $\Sigma = \{r(x)x : x \in S^{n-1}\}$ is parameterized by radial function $r(x)$, then there exist constants $0 < r_0 < R_0$ such that $r_0 \leq r(x) \leq R_0$ for $x \in S^{n-1}$. Moreover, the norm $\|\cdot\|$ is just the Minkowski functional M_Σ of Σ defined by $M_\Sigma(\lambda x) = \lambda/r(x)$ for $\lambda \geq 0$, $x \in S^{n-1}$. Note that M_Σ can also be defined through $M_\Sigma(x) = 1/\sup\{t > 0 : tx \in \mathbf{B}\}$ for $x \in \mathbb{R}^n$. Geometrically, the graph of M_Σ is the cone in $\mathbb{R}^{n+1}$ generated by (0,0) and $\Sigma \times \{1\}$.

It is easy to show the following.

Lemma 2.1. *Σ is strictly convex if and only if the strict triangle inequality holds, i.e., $\|x + y\| < \|x\| + \|y\|$, if x, $y \neq 0$ and $x/\|x\| \neq y/\|y\|$.*

We now discuss notions of support function, support set, and support map. Given $\nu \in S^{n-1}$, let T_ν be a supporting hyperplane to Σ at some $x_0 \in \Sigma$ with the outward normal ν. The function $\varphi(\nu) = \nu x_0 = \sup_{x\in\Sigma} \nu x$ is called the support function of Σ. Geometrically, $\varphi(\nu)$ denotes the distance from O to the supporting hyperplane T_ν. Obviously, the equation of T_ν can be written as $\nu x = \varphi(\nu)$. The set $T_\nu \cap \Sigma$ is called the support set of Σ with respect to T_ν or simply ν. The set-valued mapping $\Phi : \nu \longrightarrow T_\nu \cap \Sigma$ is called the support map of Σ.

We turn to the dual norm $\|\cdot\|_*$ of the norm $\|\cdot\|$. Since $\mathbb{R}^n$ is finite-dimensional, any bounded linear functional L on $(\mathbb{R}^n, \|\cdot\|)$ corresponds to a unique $\lambda\nu$ where $\lambda \geq 0$ and $\nu \in S^{n-1}$ such that $L(x) = L_{\lambda\nu}(x) = \lambda\nu x$. Let $\|\lambda\nu\|_* = \|L_{\lambda\nu}\| = \sup_{\|x\|=1} |L_{\lambda\nu}(x)|$. $\|\cdot\|_*$ is called the dual norm of $\|\cdot\|$. Since Σ is symmetric with respect to O, $\|\lambda\nu\|_* = \lambda\varphi(\nu)$. Thus, we have

Lemma 2.2. *The unit dual norm sphere is $\Sigma^* = \{y : \|y\|_* = 1\} = \{\nu/\varphi(\nu) : \nu \in S^{n-1}\}$, where $\varphi(\nu)$ is the support function of Σ.*

The following lemma follows from the definitions of the support function and support map.

Lemma 2.3. *Let $x_0 \in \Sigma$ and $\nu_0^* = \nu_0/\varphi(\nu_0) \in \Sigma^*$ with $\nu_0 \in S^{n-1}$. Then*

(i) $|\nu_0^* x_0| \leq 1$.
(ii) $\nu_0^* x_0 = 1$ *if and only if* $x_0 \in \Phi(\nu_0)$.

We say that Σ has a continuous normal field $\nu(x)$ if $\nu(x) : \Sigma \longrightarrow S^{n-1}$ is continuous and $\nu(x)$ is the unit outward normal of a supporting hyperplane $T_{\nu(x)}$ of Σ at x. By the proof of [CGH05, Theorem 6.2], if Σ has a continuous normal field $\nu(x)$, then Σ has a tangent plane at each point.

Lemma 2.4. *Σ has a continuous normal field $\nu(x)$ if and only if the Minkowski functional $M_\Sigma \in C^1(\mathbb{R}^n \setminus \{0\})$. Moreover, $\nabla M_\Sigma(x) = \nu(x)/\varphi(\nu(x))$ for $x \in \Sigma$.*

Proof. First show sufficient condition. Suppose $M_\Sigma \in C^1(\mathbb{R}^n \setminus \{0\})$. Since Σ is represented by the equation $\|x\| = M_\Sigma(x) = 1$, $\nabla M_\Sigma(x)$ is a normal of the tangent plane of Σ at x.

The homogeneity of M_Σ implies that $x\nabla M_\Sigma(x) = 1$ for $x \in \Sigma$ and $\nabla M_\Sigma(x)$ is an outward normal. Let $\nu(x) = \nabla M_\Sigma(x)/|\nabla M_\Sigma(x)|$. Obviously, $\nabla M_\Sigma(x) = \nu(x)/\varphi(\nu(x))$ and $\nu(x)$ is a continuous normal field of Σ.

Now assume that Σ has a continuous normal field $\nu(x)$. Since M_Σ has homogeneous degree one, to show $M_\Sigma \in C^1(\mathbb{R}^n \setminus \{0\})$, it suffices to prove that ∇M_Σ exists and is continuous on Σ. Let $x = r(x_0)x_0 \in \Sigma$, where $x_0 \in S^{n-1}$ and r is the radial function of Σ. We first prove

Claim: $M_\Sigma(x+h) - M_\Sigma(x) = \dfrac{\nu(x)}{\varphi(\nu(x))}h + o(|h|)$.

To proceed, write $x+h = y = |y|y_0$ with $y_0 \in S^{n-1}$, and let $z = r(y_0)y_0 \in \Sigma$. For small $|h|$, by the inner product

$$\frac{1}{2}|x_0 - y_0|^2 = 1 - x_0y_0 = \frac{|h|^2 - (|y| - |x|)^2}{2|x||y|} \le C|h|^2, \tag{2-1}$$

where C is a constant, and C will denote various constants from now onward.

Because the supporting plane of Σ at x has the equation $X\nu(x) = \varphi(\nu(x))$, $r(x_0) - r(y_0) \ge \varphi(\nu(x))/x_0\nu(x) - \varphi(\nu(x))/y_0\nu(x)$. It implies that r is Lipschitz on S^{n-1}. Therefore, $x_0r(x_0)$ is a Lipschitz mapping on S^{n-1}. By (2-1), we obtain $|x-z| \le C|x_0-y_0| \le Ch$. Since $X\nu(x) = \varphi(\nu(x))$ represents the tangent plane of Σ at x and $|z-x| \le Ch$, one gets $z\nu(x) = \varphi(\nu(x)) - o(|h|)$. We then have $(y-z)\nu(x) = h\nu(x) + o(|h|)$. Note that y and z have the same direction y_0. It follows that

$$y = \left(1 + \frac{|y| - |z|}{|z|}\right)|z|y_0 = \left(1 + \frac{h\nu(x)}{\varphi(\nu(x))} + o(|h|)\right)z.$$

By the homogeneity of M_Σ

$$M_\Sigma(y) - M_\Sigma(x) = \left(1 + \frac{h\nu(x)}{\varphi(\nu(x))} + o(|h|)\right)M_\Sigma(z) - 1 = \frac{h\nu(x)}{\varphi(\nu(x))} + o(|h|).$$

The lemma follows from the claim and that $\nu(x)$ and $\varphi(\nu(x))$ are continuous. □

Lemma 2.5. *If Σ is C^1, then Σ^* is strictly convex.*

Proof. Let $\nu_1^* = \nu_1/\varphi(\nu_1)$, $\nu_2^* = \nu_2/\varphi(\nu_2) \in \Sigma^*$, and $0 < \theta < 1$, where $\nu_1 \ne \nu_2 \in S^{n-1}$. Let $\nu^* = \theta\nu_1^* + (1-\theta)\nu_2^*$. Now show that $\|\nu^*\|_* = \sup_{x\in\Sigma}|\nu^*x| < 1$ by contradiction. Otherwise, one may assume that there exists $x_0 \in \Sigma$ such that $\nu^*x_0 = 1$. Thus, $\nu_1^*x_0 = 1$ and $\nu_2^*x_0 = 1$. By Lemma 2.3, one concludes that Σ has two different outward normals ν_1 and ν_2 at x_0, which contradicts with that Σ is C^1. □

Lemma 2.6. *$x_0 \in \Sigma$ is an outward normal of Σ^* at $\nu_0^* = \nu_0/\varphi(\nu_0)$ with $\nu_0 \in S^{n-1}$ if and only if $x_0 \in \Phi(\nu_0)$.*

Proof. Let $x_0 \in \Sigma$ be an outward normal of Σ^* at ν_0^*. Obviously, $x_0\nu_0^* = \sup_{\nu^*\in\Sigma^*} x_0\nu^* \ge x_0\nu_1^* = 1$, where $\nu_1^* \in \Sigma^*$ is an outward normal of some supporting hyperplane of Σ at x_0. By Lemma 2.3, $x_0 \in \Phi(\nu_0)$. Conversely, if $x_0 \in \Phi(\nu_0)$, then $x_0\nu_0^* = 1 = \sup_{\nu^*\in\Sigma^*} x_0\nu^*$. Hence, $x_0Y = 1$ represents a supporting hyperplane of Σ^* at ν_0^*. □

Lemma 2.7. *If Σ is strictly convex, then Σ^* is C^1.*

Proof. Since Σ is strictly convex, the support map Φ of Σ is single-valued. It is easy to show by Lemma 2.3 that $\Phi(\nu)$ is continuous on S^{n-1}. From Lemma 2.6, $\Phi(\nu^*/|\nu^*|)/|\Phi(\nu^*/|\nu^*|)|$ is a continuous normal field on Σ^*. By Lemma 2.4, $M_{\Sigma^*} \in C^1(\mathbb{R}^n \setminus \{0\})$ and its level set Σ^* is C^1. $\square$

Note that $\varphi(\nu) = M_{\Sigma^*}(\nu)$ for $\nu \in S^{n-1}$. We have

Corollary 2.8. *If Σ is strictly convex, then the support function $\varphi(\nu) \in C^1(S^{n-1})$.*

By above lemmas, if Σ is C^1 and strictly convex, $\nabla M_\Sigma(x_0) \in \Sigma^*$ is an outward normal of Σ at $x_0 \in \Sigma$, and it gives rise to a homeomorphism from Σ onto Σ^*. We call ∇M_Σ as the dual map from Σ to Σ^*. For simplicity, set $\mathcal{D}(\nu^*) = (\nabla M_\Sigma)^{-1}(\nu^*)$. Obviously, $\Phi(\nu) = \mathcal{D}(\nu^*) = \nabla M_{\Sigma^*}(\nu^*)$ for $\nu^* = \nu/\varphi(\nu) \in \Sigma^*$.

We now recall the concepts of normal section and normal curvature. If $\Sigma \in C^2$, given $x_0 \in \Sigma$ and a unit tangent vector τ at x_0, the convex curve $\mathcal{C}_\tau$, cut from Σ by the two-dimensional plane generated by the normal $\nabla M_\Sigma(x_0)$ and τ, is called a normal section. The curvature of $\mathcal{C}_\tau$ at x_0 is called the normal curvature of Σ at x_0 in the direction τ.

Σ is said to be strongly convex if $\Sigma \in C^2$ and it has strictly positive normal curvature at any point in any direction.

Finally, we recall that if Σ is strongly convex, then M_Σ and M_{Σ^*} are $C^2(\mathbb{R}^n \setminus \{0\})$ and equivalently ∇M_Σ is a C^1 diffeomorphism from Σ to Σ^*.

In the rest of the paper, we always assume that Σ is C^1 and strictly convex.

§3. Σ-PARABOLOIDS AND Σ-REFLECTORS

In this section, we first discuss distance from a point to a hyperplane in $(\mathbb{R}^n, \ \|\cdot\|)$ with C^1 strictly convex unit norm sphere Σ and geodesic realizing it. Then we introduce the notions of Σ-paraboloid and Σ-reflector. Basic properties of Σ-paraboloids including important reflection property are discussed.

Given $m^* = m/\varphi(m) \in \Sigma^*$ with $m \in S^{n-1}$ and a real number b, let $\Pi_{m^*,b}$ be the hyperplane in $\mathbb{R}^n$ with the equation

$$m^* x + b = 0 \qquad \text{or} \qquad mx + b\varphi(m) = 0. \tag{3-1}$$

The distance from a point y_0 to $\Pi_{m^*,b}$ is given by

$$dist(y_0, \Pi_{m^*,b}) = \inf_{x \in \Pi_{m^*,b}} \|x - y_0\|.$$

This distance is invariant under translations along directions perpendicular to m and has similarity property under translations along the direction of m. Indeed, by definition, it is easy to verify that $dist(y_0, \Pi_{m^*,b}) = dist(z_0, \Pi_{m^*,b})$ if $(y_0 - z_0) \perp m$. Moreover, for $\lambda > 0$, $dist(\lambda m, \Pi_{m^*}) = \lambda\, dist(m, \Pi_{m^*})$, where $\Pi_{m^*} = \Pi_{m^*,0}$.

Note that $\Pi_{m^*,1}$ with the equation $mx + \varphi(m) = 0$ is the tangent plane to Σ at $-\Phi(m)$. Because Σ is the unit norm sphere and is strictly convex, $dist(O, \Pi_{m^*,1}) = 1 = \| - \Phi(m)\|$ and the segment connecting O and $-\Phi(m) = -\mathcal{D}(m^*)$ is the only geodesic realizing the distance $dist(O, \Pi_{m^*,1})$. In general, we can prove the following.

Lemma 3.1. *Given $y_0 \in \mathbb{R}^n$ and a hyperplane $\Pi_{m^*,b}$ with equation (3-1), there exists a unique $x_0 \in \Pi_{m^*,b}$ such that $\|y_0 - x_0\| = dist(y_0, \Pi_{m^*,b})$. Furthermore, $y_0 - x_0 = (m^* y_0 + b)\Phi(m)$ and $dist(y_0, \Pi_{m^*,b}) = |m^* y_0 + b|$.*

Proof. The existence of x_0 realizing the distance $dist(y_0, \Pi_{m^*,b})$ follows from that $\mathbb{R}^n$ is of finite dimension. By the strict triangle inequality (Lemma 2.1), it is easy to prove the uniqueness of x_0. Obviously

$$\begin{aligned} dist(y_0, \Pi_{m^*,b}) &= \inf_{x \in \Pi_{m^*,b}} \|x - (my_0)m\| \\ &= \inf_{y \in \Pi_{m^*, m^* y_0 + b}} \|y\| \\ &= \inf_{z \in \Pi_{m^*,1}} |m^* y_0 + b| \, \|z\| = |m^* y_0 + b|. \end{aligned}$$

We also note that $x_0 = y_0 - (m^* y_0 + b)\Phi(m) \in \Pi_{m^*,b}$ and $\|x_0 - y_0\| = |m^* y_0 + b|$. Thus, the proof is finished. $\square$

Definition 3.2. For $m^* \in \Sigma^*$ and $b > 0$, Σ-paraboloid $P(m^*, b)$ with focus O and directrix $\Pi_{m^*,b}$ of equation (3-1) is defined by

$$P(m^*, b) = \{x \in \mathbb{R}^n : \|x\| = dist(x, \Pi_{m^*,b})\}.$$

$\Phi(m) = \mathcal{D}(m^*)$ is called the axial direction of $P(m^*, b)$.

Now discuss some properties of Σ-paraboloids.

We note that $dist(x, \Pi_{m^*,b}) = m^* x + b$ for $x \in P(m^*, b)$. By Lemma 3.1, it suffices to show $m^* x + b > 0$. Otherwise, assume $-m^* x \geq b > 0$. Then $0 < \lambda = -b/(m^* x) \leq 1$ and $x_0 = \lambda x \in \Pi_{m^*,b}$. One obtains a contradiction that $\|x\| > \|x - x_0\| \geq dist(x, \Pi_{m^*,b})$.

For $X \in P(m^*, b)$, use the (generalized) polar coordinates to write $X = \rho x$ where $x \in \Sigma$ and $\rho = \|X\|$. Clearly, $\|X\| = m^* X + b$ is equivalent to $\rho = \frac{b}{1 - m^* x}$. Therefore, $P(m^*, b)$ can be represented by the (generalized) radial function $\rho(x) = \frac{b}{1 - m^* x}$, i.e.,

$$P(m^*, b) = \left\{ \rho(x)x : x \in \Sigma \setminus \{\mathcal{D}(m^*)\} \text{ and } \rho(x) = \frac{b}{1 - m^* x} \right\}. \tag{3-2}$$

It is convenient to set $\rho(\mathcal{D}(m^*)) = \infty$. By the strict triangle inequality, $P(m^*, b)$ is strictly convex.

Lemma 3.3. *Any line ℓ with the direction $\mathcal{D}(m^*)$ intersects $P(m^*, b)$ at one and only one point. $P(m^*, b)$ can be viewed as an entire graph in the axial direction $\mathcal{D}(m^*)$.*

Proof. First show that $\ell \cap P(m^*, b)$ contains at most one point. Obviously, if $0 \in \ell$, then $\ell \cap P(m^*, b) = \{-b\mathcal{D}(m^*)/2\}$. It remains to consider the case that $0 \notin \ell$. We proceed by a contradiction argument and assume that there exist X_1, $X_2 \in \ell \cap P(m^*, b)$ such that $X_2 - X_1 = \lambda \mathcal{D}(m^*)$ with $\lambda > 0$. Let Y_0 be the intersecting point of ℓ and $\Pi_{m^*,b}$. Then $X_1 - Y_0 = \tau \mathcal{D}(m^*)$ where $\tau = m^* X_1 + b > 0$, and $X_2 - Y_0 = (\lambda + \tau)\mathcal{D}(m^*)$.

Since X_1, $X_2 \in P(m^*, b)$ and by Lemma 3.1, $\|X_1\| = \tau$ and $\|X_2\| = \lambda + \tau$. Therefore, $\|X_2\| = \|X_1\| + \|X_2 - X_1\|$. It contradicts with the strict triangle inequality, because X_1 is not parallel to $\mathcal{D}(m^*)$.

To finish the proof of the lemma, it suffices to show $\ell \cap P(m^*, b) \neq \emptyset$. For $t \geq 0$, let $Y_t = Y_0 + t\mathcal{D}(m^*)$. Obviously, $Y_t \in \ell$. We claim

$$\|Y_t\| = t - b + o(1), \qquad \text{as } t \longrightarrow \infty. \tag{3-3}$$

To prove (3-3), note $\|Y_t\| = M_\Sigma(Y_t) = tM_\Sigma(\mathcal{D}(m^*) + Y_0/t)$. Since $M_\Sigma \in C^1(\mathbb{R}^n \setminus \{0\})$, for large t, we have

$$\begin{aligned} M_\Sigma(\mathcal{D}(m^*) + Y_0/t) &= M_\Sigma(\mathcal{D}(m^*)) + \nabla M_\Sigma(\mathcal{D}(m^*))Y_0/t + o(|Y_0|/t) \\ &= 1 + m^*Y_0/t + o(1/t) \\ &= 1 - b/t + o(1/t). \end{aligned}$$

Thus, (3-3) follows. Since $m^*Y_0 + b = 0$, it is easy to verify that $H(t) = \|Y_t\| - (m^*Y_t + b)$ is continuous on $[0, \infty)$, $H(0) = \|Y_0\| > 0$, and $H(t) = -b + o(1) < 0$ as $t \longrightarrow \infty$. One concludes that there exists some $t_0 > 0$ such that $H(t_0) = 0$ and $Y_{t_0} \in \ell \cap P(m^*, b)$. □

Now turn to the reflection property of Σ-paraboloids.

Lemma 3.4. *Let Y_0 be inside $P(m^*, b)$ and X_0 be the unique point on $P(m^*, b)$ such that $Y_0 - X_0$ is in the axial direction $\mathcal{D}(m^*)$. Then for any $X \in P(m^*, b) \setminus \{X_0\}$*

$$\|X_0\| + \|Y_0 - X_0\| < \|X\| + \|Y_0 - X\|. \tag{3-4}$$

Consequently, the path OX_0Y_0 is the minimizing path among paths OXY_0 with $X \in P(m^*, b)$.

Proof. Let X_0', $X' \in \Pi_{m^*,b}$ such that $dist(X_0, \Pi_{m^*,b}) = \|X_0 - X_0'\|$ and $dist(X, \Pi_{m^*,b}) = \|X - X'\|$, where $\Pi_{m^*,b}$ is the directrix of $P(m^*, b)$. By Lemma 3.1, $X - X'$ is parallel to $X_0 - X_0'$ and hence $X' \neq X_0'$. Since $X_0 \in P(m^*, b)$, one obtains

$$\|X_0\| + \|Y_0 - X_0\| = \|X_0 - X_0'\| + \|Y_0 - X_0\| = dist(Y_0, \Pi_{m^*,b}).$$

By Lemma 3.1

$$dist(Y_0, \Pi_{m^*,b}) < \|Y_0 - X'\| < \|X - X'\| + \|Y_0 - X\|.$$

Because $X \in P(m^*, b)$, we conclude (3-4). □

We remark that (3-4) also holds for any point $X \neq X_0$ on the tangent plane to $P(m^*, b)$ at X_0. The Euler equation for (3-4) is equivalent to that $\nabla M_\Sigma(X_0) - \nabla M_\Sigma(Y_0 - X_0)$ is a normal of the tangent plane to $P(m^*, b)$ at X_0. Therefore, according to the Fermat principle, a light ray emitted from O incident on $X_0 \in P(m^*, b)$ is reflected in the axial direction $\mathcal{D}(m^*)$.

Definition 3.5. A closed surface $\mathcal{R}$ in $\mathbb{R}^n$ is a Σ-reflector (or a reflector for simplicity) if for any $X_0 \in \mathcal{R}$ there exists a Σ-paraboloid $P(m^*, b)$ such that $X_0 \in P(m^*, b)$ and $\mathcal{R}$ is contained in the convex body limited by $P(m^*, b)$. $P(m^*, b)$ is called a supporting Σ-paraboloid of $\mathcal{R}$ at X_0.

By definition, any Σ-reflector is strictly convex.

Lemma 3.6. *Let $\mathcal{R}$ be a Σ-reflector. If $\mathcal{R}$ has two distinct supporting Σ-paraboloids at X_0, then X_0 is a singular point of $\mathcal{R}$, i.e., $\mathcal{R}$ has no tangent plane at X_0.*

Proof. Assume that $P(m_1^*, b_1)$ and $P(m_2^*, b_2)$ are two distinct supporting Σ-paraboloids to $\mathcal{R}$ at X_0. Obviously, $m_1^* \neq m_2^*$. For $i = 1$, 2, $\nabla M_\Sigma(X_0) - m_i^*$ is a normal of the tangent plane to $P(m_i^*, b_i)$ at X_0 which is also a supporting hyperplane to $\mathcal{R}$. $\nabla M_\Sigma(X_0) \neq m_i^*$ for $i = 1$, 2. Otherwise, X_0 must be in the axial direction $\mathcal{D}(m_i^*)$. Since $\nabla M_\Sigma(X_0)$, m_1^*, and m_2^* are three distinct points on Σ^*, and Σ^* is strictly convex, $\nabla M_\Sigma(X_0) - m_1^*$ and $\nabla M_\Sigma(X_0) - m_2^*$ are not parallel. Therefore, $\mathcal{R}$ has two different supporting hyperplanes at X_0 and has no tangent plane at X_0. $\square$

§4. Weak Solutions

The purpose in this section is to discuss Σ-front map, Σ-reflector map, Σ-visibility map, Radon measures associated with these maps, notion of weak solution to the Σ-reflector problem, and Legendre transform.

Let $\mathcal{R}$ be a Σ-reflector parameteried by $X = \rho(x)x$ for $x \in \Sigma$. Since $\mathcal{R}$ is strictly convex and encloses O, $\rho(x)$ is strictly positive, bounded, and Lipschitz on Σ.

Definition 4.1. For $y \in \Sigma$, the Σ-front map associated with $\mathcal{R}$ is

$$\mathcal{F}_\mathcal{R}(y) = \{m^* \in \Sigma^* : P(m^*, b) \text{ supports } \mathcal{R} \text{ at } \rho(y)y \text{ for some } b > 0\}.$$

For $E \subset \Sigma$, set $\mathcal{F}_\mathcal{R}(E) = \cup_{y \in E} \mathcal{F}_\mathcal{R}(y)$.

Definition 4.2. For $y \in \Sigma$, the Σ-reflector map associated with $\mathcal{R}$ is

$$\mathcal{N}_\mathcal{R}(y) = \mathcal{D}[\mathcal{F}_\mathcal{R}(y)] = \{\mathcal{D}(m^*) \in \Sigma : P(m^*, b) \text{ supports } \mathcal{R} \text{ at } \rho(y)y \text{ for some } b > 0\}.$$

For $E \subset \Sigma$, set $\mathcal{N}_\mathcal{R}(E) = \cup_{y \in E} \mathcal{N}_\mathcal{R}(y)$.

Definition 4.3. For $\gamma \in \Sigma$, the Σ-visibility map associated with $\mathcal{R}$ is

$$\begin{aligned} \mathcal{V}_\mathcal{R}(\gamma) &= \{y \in \Sigma : \gamma \in \mathcal{N}_\mathcal{R}(y)\} \\ &= \{y \in \Sigma : P(\nabla M_\Sigma(\gamma), b) \text{ supports } \mathcal{R} \text{ at } \rho(y)y \text{ for some } b > 0\}. \end{aligned}$$

For $E \subset \Sigma$, set $\mathcal{V}_\mathcal{R}(E) = \cup_{\gamma \in E} \mathcal{V}_\mathcal{R}(\gamma)$.

We note that $\mathcal{F}_\mathcal{R}$, $\mathcal{N}_\mathcal{R}$, $\mathcal{V}_\mathcal{R}$ are surjective set-valued mappings. To see surjectivity of $\mathcal{F}_\mathcal{R}$, given $m^* \in \Sigma^*$, let

$$b_0 = \inf\{b > 0 : \rho(x) \leq b/(1 - m^* x) \text{ for } x \in \Sigma\}.$$

Then $P(m^*, b_0)$ is a supporting Σ-paraboloid to $\mathcal{R}$. The surjectivity of $\mathcal{N}_\mathcal{R}$ and $\mathcal{V}_\mathcal{R}$ follows.

Lemma 4.4. *$\mathcal{C} = \{E \subset \Sigma : \mathcal{V}_{\mathcal{R}}(E)$ is Lebesgue measurable$\}$ is a σ-algebra containing all Borel sets in Σ.*

Proof. Obviously, $\mathcal{V}_{\mathcal{R}}(\Sigma) = \Sigma$ and $\mathcal{V}_{\mathcal{R}}(\emptyset) = \emptyset$. By definition, $\mathcal{V}_{\mathcal{R}}(\cup_{i=1}^{\infty} E_i) = \cup_{i=1}^{\infty} \mathcal{V}_{\mathcal{R}}(E_i)$. Therefore, if $E_i \in \mathcal{C}$, for $i \geq 1$, then $\cup_{i=1}^{\infty} E_i \in \mathcal{C}$. Clearly, for $E \subset \Sigma$

$$\begin{aligned} \mathcal{V}_{\mathcal{R}}(E^c) &= \{y \in \Sigma : \mathcal{N}_{\mathcal{R}}(y) \cap E^c \neq \emptyset\} \\ &= \{y \in \Sigma : \mathcal{N}_{\mathcal{R}}(y) \cap E = \emptyset\} \cup \{y \in \Sigma : \mathcal{N}_{\mathcal{R}}(y) \cap E^c \neq \emptyset,\ \mathcal{N}_{\mathcal{R}}(y) \cap E \neq \emptyset\} \\ &= [\mathcal{V}_{\mathcal{R}}(E)]^c \cup [\mathcal{V}_{\mathcal{R}}(E^c) \cap \mathcal{V}_{\mathcal{R}}(E)]. \end{aligned} \tag{4-1}$$

If $y \in \mathcal{V}_{\mathcal{R}}(E^c) \cap \mathcal{V}_{\mathcal{R}}(E)$, $\mathcal{R}$ has at least two distinct supporting Σ-paraboloids at $\rho(y)y$. By Lemma 3.6, $\rho(y)y$ is a singular point of $\mathcal{R}$ and the area measure of $\mathcal{V}_{\mathcal{R}}(E^c) \cap \mathcal{V}_{\mathcal{R}}(E)$ is 0. It follows that if $E \in \mathcal{C}$ then $E^c \in \mathcal{C}$. We have proved that $\mathcal{C}$ is a σ-algebra.

To prove that $\mathcal{C}$ contains all Borel subsets, it suffices to show that $\mathcal{V}_{\mathcal{R}}(K)$ is compact if K is compact. Let $y_i \in \mathcal{V}_{\mathcal{R}}(K)$ for $i \geq 1$. There exists $\gamma_i \in \mathcal{N}_{\mathcal{R}}(y_i) \cap K$. Let $P(m_i^*, b_i)$ be the supporting Σ-paraboloid to $\mathcal{R}$ at $\rho(y_i)y_i$, where $m_i^* = \nabla M_{\Sigma}(\gamma_i)$. We have

$$\rho(x)(1 - m_i^* x) \leq b_i \qquad \text{for } x \in \Sigma, \tag{4-2}$$

where the equality in (4-2) occurs at $x = y_i$, $\rho(x)$ is the generalized radial function of $\mathcal{R}$, and $a_1 \leq \rho(x) \leq a_2$ on Σ for some constants $a_2 \geq a_1 > 0$. Assume through subsequence that $y_i \longrightarrow y_0$, $\gamma_i \longrightarrow \gamma_0 \in K$, $m_i^* \longrightarrow m_0^* = \nabla M_{\Sigma}(\gamma_0)$, as $i \longrightarrow \infty$. It is easy to see $b_i \geq 2a_1$ by choosing $x = -\gamma_i$ in (4-2). Clearly $b_i \leq 2a_2$. So one may assume $\lim_{i\to\infty} b_i = b_0$. By taking limit in (4-2), $\rho(x)(1 - m_0^* x) \leq b_0$ for $x \in \Sigma$ and the equality occurs at $x = y_0$. Therefore, $P(m_0^*, b_0)$ is a supporting Σ-paraboloid to $\mathcal{R}$ at $\rho(y_0)y_0$ and $y_0 \in \mathcal{V}_{\mathcal{R}}(\gamma_0)$. This proves $\mathcal{V}_{\mathcal{R}}(K)$ is compact. □

Lemma 4.5. *Given a nonnegative $f \in L^1(\Sigma)$, the set function*

$$\mu_{f,\mathcal{R}}(E) = \int_{\mathcal{V}_{\mathcal{R}}(E)} f\, dx$$

is a finite Borel measure defined on $\mathcal{C}$, where dx donotes the area measure on Σ.

Proof. Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of pairwise disjont sets in $\mathcal{C}$. Let $H_1 = \mathcal{V}_{\mathcal{R}}(E_1)$, and $H_k = \mathcal{V}_{\mathcal{R}}(E_k) \setminus \cup_{i=1}^{k-1} \mathcal{V}_{\mathcal{R}}(E_i)$, for $k \geq 2$. Because $H_i \cap H_j = \emptyset$ for $i \neq j$ and $\cup_{k=1}^{\infty} H_k = \cup_{k=1}^{\infty} \mathcal{V}_{\mathcal{R}}(E_k)$, it is easy to get

$$\mu_{f,\mathcal{R}}(\cup_{k=1}^{\infty} E_k) = \int_{\cup_{k=1}^{\infty} H_k} f\, dx = \sum_{k=1}^{\infty} \int_{H_k} f\, dx.$$

Observe that $\mathcal{V}_{\mathcal{R}}(E_k) - H_k = \mathcal{V}_{\mathcal{R}}(E_k) \cap (\cup_{i=1}^{k-1} \mathcal{V}_{\mathcal{R}}(E_i))$ is a subset of the singular set of $\mathcal{R}$ and has area measure 0 for $k \geq 2$. Therefore, $\int_{H_k} f\, dx = \mu_{f,\mathcal{R}}(E_k)$ and the σ-additivity of $\mu_{f,\mathcal{R}}$ follows. □

The Σ-visibility map possesses some weak compactness under uniform convergence of Σ-reflectors.

Lemma 4.6. *Let $\mathcal{R}_j = \{\rho_j(x)x : x \in \Sigma\}$, $j \geq 1$, be Σ-reflectors. Suppose that $0 < a_1 \leq \rho_j \leq a_2$ and $\rho_j \longrightarrow \rho$ uniformly on Σ. Then $\mathcal{R} = \{\rho(x)x : x \in \Sigma\}$ is a Σ-reflector. Furthermore:*

(i) *For any compact set $K \subset \Sigma$, $\limsup_{j\to\infty} \mathcal{V}_{\mathcal{R}_j}(K) \subset \mathcal{V}_{\mathcal{R}}(K)$.*

(ii) *For any open set $G \subset \Sigma$, $\mathcal{V}_{\mathcal{R}}(G) \subset \liminf_{j\to\infty} \mathcal{V}_{\mathcal{R}_j}(G) \cup S$, where $S = \{x \in \Sigma : \mathcal{R}$ has no tangent plane at $\rho(x)x\}$ is the singular set of $\mathcal{R}$ with the area measure 0.*

Proof. It is easy to verify that $\mathcal{R}$ is a Σ-reflector. The proof of (i) is similar to that of Lemma 4.4. Let $y \in \limsup_{j\to\infty} \mathcal{V}_{\mathcal{R}_j}(K)$. Without loss of generality, may assume that $y \in \mathcal{V}_{\mathcal{R}_j}(K)$ for all $j \geq 1$. Then there exists $\gamma_j \in \mathcal{N}_{\mathcal{R}_j}(y) \cap K$. Let $P(\nabla M_\Sigma(\gamma_j), b_j)$ support $\mathcal{R}_j$ at $\rho_j(y)y$, i.e., $\rho_j(x)(1 - \nabla M_\Sigma(\gamma_j)x) \leq b_j$ on Σ and equality occurs at $x = y$. One may assume that $\gamma_j \longrightarrow \gamma_0 \in K$ and $b_j \longrightarrow b_0 > 0$. By taking limit, we conclude that $P(\nabla M_\Sigma(\gamma_0), b_0)$ supports $\mathcal{R}$ at $\rho(y)y$. This proves that $y \in \mathcal{V}_{\mathcal{R}}(\gamma_0)$.

To prove (ii), given an open set $G \subset \Sigma$, by (i), $\limsup_{j\to\infty} \mathcal{V}_{\mathcal{R}_j}(G^c) \subset \mathcal{V}_{\mathcal{R}}(G^c)$. By (4-1), it gives rise to

$$\begin{aligned}\limsup_{j\to\infty}[\mathcal{V}_{\mathcal{R}_j}(G)]^c &\subset \limsup_{j\to\infty}\left([\mathcal{V}_{\mathcal{R}_j}(G)]^c \cup [\mathcal{V}_{\mathcal{R}_j}(G) \cap \mathcal{V}_{\mathcal{R}_j}(G^c)]\right)\\ &\subset [\mathcal{V}_{\mathcal{R}}(G)]^c \cup [\mathcal{V}_{\mathcal{R}}(G) \cap \mathcal{V}_{\mathcal{R}}(G^c)].\end{aligned}$$

By De Morgan's law

$$\mathcal{V}_{\mathcal{R}}(G) \cap [\mathcal{V}_{\mathcal{R}}(G) \cap \mathcal{V}_{\mathcal{R}}(G^c)]^c \subset \liminf_{j\to\infty} V_{\mathcal{R}_j}(G).$$

Since $\mathcal{V}_{\mathcal{R}}(G) \cap \mathcal{V}_{\mathcal{R}}(G^c) \subset S$, we conclude $\mathcal{V}_{\mathcal{R}}(G) \subset \liminf_{j\to\infty} \mathcal{V}_{\mathcal{R}_j}(G) \cup S$. $\square$

Below we define weak solutions to the Σ-reflector problem through energy conservation as [CO94].

Definition 4.7. Let $f \geq 0$ and $f \in L^1(\Sigma)$, and μ be a nonnegative Radon measure on Σ. Suppose that f and μ satisfy the compatible energy condition

$$\int_\Sigma f\,dx = \mu(\Sigma) > 0. \tag{4-3}$$

A Σ-reflector $\mathcal{R}$ is a weak solution of the Σ-reflector problem with emitted illumination intensity $f(x)$ and prescribed reflected illumination intensity μ in a medium in which light wavefronts are similar to Σ if for any Borel set $E \subset \Sigma$

$$\mu_{f,\mathcal{R}}(E) = \int_{\mathcal{V}_{\mathcal{R}}(E)} f\,dx = \mu(E). \tag{4-4}$$

Remark 4.8. If $\mathcal{R}$ is a weak solution to the Σ-reflector problem satisfying (4-4) and $\lambda > 0$, then $\lambda\mathcal{R}$, λ-dilation of $\mathcal{R}$ with respect to O, is also a weak solution to the same reflector problem.

To see this, observe that $\mathcal{N}_{\lambda\mathcal{R}}(y) = \mathcal{N}_{\mathcal{R}}(y)$ for $y \in \Sigma$. Therefore, $\mathcal{V}_{\lambda\mathcal{R}}(E) = \{y \in \Sigma : \mathcal{N}_{\lambda\mathcal{R}}(y) \cap E \neq \emptyset\} = \mathcal{V}_{\mathcal{R}}(E)$ and $\mu_{f,\lambda\mathcal{R}}(E) = \mu_{f,\mathcal{R}}(E) = \mu(E)$.

Definition 4.9. The Legendre transform of a Σ-reflector $\mathcal{R}$ parameterized by $\rho(x)$ is the Σ^*-reflector $\mathcal{R}^* = \{\rho^*(m^*)m^* : m^* \in \Sigma^*\}$ where

$$\rho^*(m^*) = \inf_{x\in\Sigma} \frac{1}{\rho(x)(1-m^*x)}.$$

Note that if $m_0^* \in \mathcal{F}_{\mathcal{R}}(x_0)$ then $x_0 \in \mathcal{F}_{\mathcal{R}^*}(m_0^*)$, which also verifies that $\mathcal{R}^*$ is a Σ^*-reflector. Indeed, if $P(m_0^*, b)$ is a supporting Σ-paraboloid to $\mathcal{R}$ at $\rho(x_0)x_0$, then $P^*(x_0, \frac{1}{\rho(x_0)}) = \left\{ \dfrac{m^*}{\rho(x_0)(1-x_0m^*)} : m^* \in \Sigma^* \right\}$ is a supporting Σ^*-paraboloid to $\mathcal{R}^*$ at $\rho^*(m_0^*)m_0^*$.

Lemma 4.10. *If $\mathcal{R}$ is a Σ-reflector and $\mathcal{R}^{**}$ denotes the Legendre transform of $\mathcal{R}^*$, then $\mathcal{R}^{**} = \mathcal{R}$.*

Proof. Given $x_0 \in \Sigma$, let $P(m_0^*, b)$ be a supporting Σ-paraboloid to $\mathcal{R}$ at $\rho(x_0)x_0$. Since $P^*(x_0, \frac{1}{\rho(x_0)})$ supports $\mathcal{R}^*$ at $\rho^*(m_0^*)m_0^*$, $\rho^*(m^*)(1-x_0m^*)$ attains the maximum $1/\rho(x_0)$ at m_0^*. We obtain $\rho^{**}(x_0) = \inf_{m^*\in\Sigma^*} 1/[\rho^*(m^*)(1-x_0m^*)] = \rho(x_0)$. $\square$

Lemma 4.11. *Assume Σ is strongly convex. Let $\mathcal{R}$ be a Σ-reflector. Then $\bigcup_{x_1\neq x_2} [\mathcal{N}_{\mathcal{R}}(x_1)\cap \mathcal{N}_{\mathcal{R}}(x_2)] \subset \Sigma$ has measure zero.*

Proof. If $\gamma_0 \in \mathcal{N}_{\mathcal{R}}(x_1) \cap \mathcal{N}_{\mathcal{R}}(x_2)$ where $x_1 \neq x_2$, then $m_0^* = \nabla M_\Sigma(\gamma_0) \in \mathcal{F}_{\mathcal{R}}(x_1) \cap \mathcal{F}_{\mathcal{R}}(x_2)$ and x_1, $x_2 \in \mathcal{F}_{\mathcal{R}^*}(m_0^*)$. One obtains that m_0^* is a singular point of $\mathcal{R}^*$ and $\nabla M_\Sigma [\cup_{x_1\neq x_2}(\mathcal{N}_{\mathcal{R}}(x_1)\cap\mathcal{N}_{\mathcal{R}}(x_2))]$ is contained in the singular set of $\mathcal{R}^*$. Since $\nabla M_\Sigma : \Sigma \longrightarrow \Sigma^*$ is a C^1-diffeomorphism, the lemma is proved. $\square$

Similar to the proofs of Lemma 4.4 and 4.5, and by Lemma 4.11, if Σ is strongly convex, then the collection of sets $E \subset \Sigma$ such that $\mathcal{N}_{\mathcal{R}}(E)$ is Lebesgue measurable is a σ-algebra containing all Borel sets in Σ, and for any nonnegative $g(\gamma) \in L^1(\Sigma)$, $\int_{\mathcal{N}_{\mathcal{R}}(E)} g(\gamma)\,d\gamma$ as a set function of E is a finite Borel measure.

Lemma 4.12. *Let f, $g \geq 0$ and f, $g \in L^1(\Sigma)$, and let $d\mu = g(\gamma)d\gamma$. Suppose that Σ is strongly convex and (4-3) holds. Then a Σ-reflector $\mathcal{R}$ is a weak solution of the Σ-reflector problem in the sense of (4-4) if and only if for any Borel set $E \subset \Sigma$*

$$\int_E f(x)\,dx = \int_{\mathcal{N}_{\mathcal{R}}(E)} g(\gamma)\,d\gamma = \int_{\mathcal{F}_{\mathcal{R}}(E)} g^*(m^*)\,dm^*,$$

where $g^*(m^*) = g(\mathcal{D}(m^*))J_{\mathcal{D}}(m^*)$ and $J_{\mathcal{D}}$ is the Jacobian of $\mathcal{D}$.

Proof. By change of variables, it suffices to prove that $\mathcal{R}$ satisfies (4-4) is equivalent to that for any Borel set $E \subset \Sigma$

$$\int_E f(x)\,dx = \int_{\mathcal{N}_{\mathcal{R}}(E)} g(\gamma)\,d\gamma. \tag{4-5}$$

Assume that (4-4) holds. Given a Borel set $E \subset \Sigma$, there exist Borel sets B_1, B_2 such that $B_1 \subset \mathcal{N}_{\mathcal{R}}(E) \subset B_2$ and $meas(B_2 \setminus B_1) = 0$. Obviously, $\mathcal{V}_{\mathcal{R}}(B_2) \supset \mathcal{V}_{\mathcal{R}}(\mathcal{N}_{\mathcal{R}}(E)) \supset E$. By (4-4), we have

$$\int_{\mathcal{N}_{\mathcal{R}}(E)} g\, d\gamma = \int_{\mathcal{V}_{\mathcal{R}}(B_2)} f\, dx \geq \int_E f\, dx.$$

On the other hand, by Lemma 4.11, the set of all γ such that $\mathcal{V}_{\mathcal{R}}(\gamma)$ contains at least two points is contained in some Borel set B_3 of measure zero. So $\mathcal{V}_{\mathcal{R}}(B_1 \setminus B_3) \subset E$. By (4-4), we obtain

$$\int_{\mathcal{N}_{\mathcal{R}}(E)} g\, d\gamma = \int_{\mathcal{V}_{\mathcal{R}}(B_1 \setminus B_3)} f\, dx \leq \int_E f\, dx.$$

Thus, (4-5) is established. Similarly, one can show that (4-5) implies (4-4). $\square$

§5 Existence of Weak Solutions

We first establish the existence of weak solution to the Σ-reflector problem for which all lights are reflected into only finite directions. In this case, weak solutions can be realized as poly-Σ-paraboloids. The proof of Theorem 5.1 is similar to that in [CO94].

Theorem 5.1. *Let $f \in L^1(\Sigma)$ and $f \geq 0$, and $\mu = \sum_{i=1}^k g_i \delta_{\gamma_i}$ where $k \geq 2$, $\{\gamma_i\}$ are distinct points on Σ, g_i is a positive constant and δ_{γ_i} is the Dirac measure at γ_i for $1 \leq i \leq k$. Suppose the energy conservation*

$$\int_\Sigma f\, dx = \sum_{i=1}^k g_i. \tag{5-1}$$

Then there exists a weak solution $\mathcal{R}$ to the Σ-reflector problem in the sense of (4-4) such that $\Sigma = \cup_{i=1}^k V_{\mathcal{R}}(\gamma_i)$ and

$$\mu_{f,\mathcal{R}}(\{\gamma_i\}) = \int_{\mathcal{V}_{\mathcal{R}}(\gamma_i)} f\, dx = g_i, \qquad \text{for } 1 \leq i \leq k. \tag{5-2}$$

Proof. Let $m_i^* = \nabla M_\Sigma(\gamma_i)$. For $b = (b_1, \cdots, b_k) \in \mathbb{R}^k$ with each $b_i > 0$, let $\mathcal{R}(b) = \mathcal{R}(b_1, \cdots, b_k)$ be the Σ-reflector consisting of pieces of $P(m_i^*, b_i)$ defined by

$$\mathcal{R}(b) = \left\{ \rho(x)x : x \in \Sigma,\ \rho(x) = \min_{1 \leq i \leq k} \frac{b_i}{1 - m_i^* x} \right\}.$$

Because any point on $\mathcal{R}(b)$ is on some $P(m_i^*, b_i)$, $\Sigma = \cup_{i=1}^k \mathcal{V}_{\mathcal{R}(b)}(\gamma_i)$. Consequently, $\int_\Sigma f\, dx = \sum_{i=1}^k \mu_{f,\mathcal{R}(b)}(\gamma_i)$, where $\mu_{f,\mathcal{R}}(\gamma_i) = \mu_{f,\mathcal{R}}(\{\gamma_i\})$ for simplicity. We note that some $P(m_i^*, b_i)$ may not support $\mathcal{R}(b)$.

Fix $b_1 = 1$ and let $W \subset \mathbb{R}^k$ be the set of all $b = (1, b_2, \cdots, b_k)$ with each $b_i > 0$ such that $\mathcal{R}(b) = \mathcal{R}(1, b_2, \cdots, b_k)$ satisfies

$$\mu_{f,\mathcal{R}(b)}(\gamma_i) \leq g_i, \qquad \text{for } i = 2, \cdots, k. \tag{5-3}$$

By definition and (5-1), if $b \in W$, then $\mu_{f,\mathcal{R}(b)}(\gamma_1) \geq g_1$. As a result, $\mathcal{V}_{\mathcal{R}(b)}(\gamma_1)$ is of positive area. Let $y \in \mathcal{V}_{\mathcal{R}(b)}(\gamma_1) \backslash S(b)$ where $S(b)$ is the singular set of $\mathcal{R}(b)$. Then $\mathcal{N}_{\mathcal{R}(b)}(y) = \{\gamma_1\}$. Therefore, $\rho(y)y \in \mathcal{R}(b) \cap P(m_1^*, 1)$ and $P(m_1^*, 1)$ is the supporting Σ-paraboloid to $\mathcal{R}(b)$ at $\rho(y)y$. To prove the theorem, we first prove four claims.

Claim 1: $W \neq \emptyset$.

To prove, choose large $b_2, \cdots, b_k$. If $\rho(x)x \in \mathcal{R}(b) \cap P(m_i^*, b_i)$ for some $i \neq 1$, then $1/(1 - m_1^* x) \geq \rho(x) \geq b_i/2$. Since b_i is large and Σ is strictly convex, x must be in some small neighborhood $N_\eta(\gamma_1) = \{\gamma \in \Sigma : |\gamma - \gamma_1| < \eta\}$. Therefore, $\mathcal{V}_{\mathcal{R}(b)}(\gamma_i) \setminus S(b) \subset N_\eta(\gamma_1)$ and $\mu_{f,\mathcal{R}(b)}(\gamma_i) \leq g_i$, for $i \neq 1$. Thus, $W \neq \emptyset$.

Claim 2: There exists some $\varepsilon > 0$ such that if $b = (1, b_2, \cdots, b_k) \in W$, then $b_i \geq \varepsilon$ for $i = 2, \cdots, k$.

To proceed, choose some small $\eta > 0$ such that $\int_{N_\eta(\gamma_i)} f\, dx \leq g_1/2$ for $i \neq 1$. Then there exists some $y \in \mathcal{V}_{\mathcal{R}(b)}(\gamma_1) \setminus [S(b) \cup N_\eta(\gamma_i)]$. Otherwise, $\mu_{f,\mathcal{R}(b)}(\gamma_1) \leq g_1/2$ which yields a contradiction. We have $\rho(y)y \in \mathcal{R}(b) \cap P(m_1^*, 1)$ and $1/(1 - m_1^* y) \leq b_i/(1 - m_i^* y)$. Since $y \notin N_\eta(\gamma_i)$, $b_i \geq \varepsilon = \varepsilon(\eta)$.

Claim 3: For $1 \leq i \leq k$, $\mu_{f,\mathcal{R}(b)}(\gamma_i)$ is a continuous function on W with respect to the Euclidean distance.

To prove, assume that $b^j = (1, b_2^j, \cdots, b_k^j) \in W$ converges to $b = (1, b_2, \cdots, b_k) \in W$. Since b_i^j is bounded away from ∞ and 0, we obtain

$$\rho_j(x) = \min_{1 \leq i \leq k} \frac{b_i^j}{1 - m_i^* x} \longrightarrow \rho(x) = \min_{1 \leq i \leq k} \frac{b_i}{1 - m_i^* x}, \qquad \text{uniformly on } \Sigma,$$

and $0 < r_1 \leq \rho_j(x) \leq r_2$ for some constants r_1, r_2. By Lemma 4.6, for $1 \leq i \leq k$

$$\overline{\lim_{j \to \infty}}\, \mu_{f,\mathcal{R}(b^j)}(\gamma_i) \leq \int_{\overline{\lim\limits_{j \to \infty}}\, \mathcal{V}_{\mathcal{R}(b^j)}(\gamma_i)} f\, dx \leq \int_{\mathcal{V}_{\mathcal{R}(b)}(\gamma_i)} f\, dx = \mu_{f,\mathcal{R}(b)}(\gamma_i). \tag{5-4}$$

Similarly, for any open set $G \subset \Sigma$, we have

$$\underline{\lim_{j \to \infty}} \int_{\mathcal{V}_{\mathcal{R}(b^j)}(G)} f\, dx \geq \int_{\mathcal{V}_{\mathcal{R}(b)}(G)} f\, dx.$$

Choose a small neighborhood G of γ_i such that $\gamma_l \notin G$ for $l \neq i$. Obviously, $\mathcal{V}_{\mathcal{R}(b^j)}(G \setminus \{\gamma_i\}) \subset S(b^j)$. It follows that

$$\underline{\lim_{j \to \infty}} \int_{\mathcal{V}_{\mathcal{R}(b^j)}(\gamma_i)} f\, dx \geq \int_{\mathcal{V}_{\mathcal{R}(b)}(\gamma_i)} f\, dx,$$

which, together with (5-4), implies the continuity of $\mu_{f,\mathcal{R}(b)}(\gamma_i)$.

Now pick up $b^0 = (1, b_2^0, \cdots, b_k^0) \in W$ and fix it. Define

$$W_0 = \{b \in W : b_1 = 1 \text{ and } b_i \leq b_i^0 \text{ for } i \neq 1\}.$$

Claim 4: W_0 is compact and $W_0 \subset \{1\} \times [\varepsilon, b_2^0] \times \cdots \times [\varepsilon, b_k^0]$ where ε is the constant given in claim 2.

It suffices to prove that W_0 is closed. Given that $b^j \in W_0$ and b^j converges to $b = (1, b_2, \cdots, b_k)$, obviously $b_i \le b_i^0$ for $2 \le i \le k$. Since (5-3) holds for $\mathcal{R}(b^j)$ and $\mu_{f,\mathcal{R}(b)}(\gamma_i)$ is continuous, $\mathcal{R}(b)$ also satisfies (5-3) and $b \in W_0$.

Now turn to the proof of the theorem. Let $d(b) = \sum_{i=1}^k b_i$ for $b = (1, b_2, \cdots, b_k) \in W_0$. By compactness of W_0, $d(b)$ must attain the minimum on W_0 at some $b^* = (1, b_2^*, \cdots, b_k^*)$. We use a contradiction argument to show that $\mu_{f,\mathcal{R}(b^*)}(\gamma_i) = g_i$ for $2 \le i \le k$ and hence $\mathcal{R}(b^*)$ is a weak solution satisfying (5-2). May assume that $\mu_{f,\mathcal{R}(b^*)}(\gamma_2) < g_2$. Consider $b_\theta^* = (1, \theta b_2^*, b_3^*, \cdots, b_k^*)$ where $0 < \theta < 1$. It is easy to verify that

$$\mathcal{V}_{\mathcal{R}(b_\theta^*)}(\gamma_i) \setminus S(b_\theta^*) \subset \mathcal{V}_{\mathcal{R}(b^*)}(\gamma_i), \quad i \ne 2. \tag{5-5}$$

If θ is close to 1, by the continuity of $\mu_{f,\mathcal{R}(b)}(\gamma_2)$, $\mu_{f,\mathcal{R}(b_\theta^*)}(\gamma_2) < g_2$. By (5-5), $\mu_{f,\mathcal{R}(b_\theta^*)}(\gamma_i) \le \mu_{f,\mathcal{R}(b^*)}(\gamma_i) \le g_i$ for $i \ge 3$. Therefore, $b_\theta^* \in W_0$. However, $d(b_\theta^*) < d(b^*)$. That is a contradiction. Thus, the proof of the theorem is complete. $\square$

Remark 5.2. If $\int_\Sigma f\, dx = g_0 > 0$ and $\inf_\Sigma f > 0$, then there exists no (closed) Σ-reflector $\mathcal{R}$ satisfying $\mu_{f,\mathcal{R}} = g_0 \delta_{\gamma_0}$, where δ_{γ_0} is the Dirac measure at γ_0.

To prove, assume by contradiction that there exists a solution $\mathcal{R}$. Observe that $\mathcal{V}_\mathcal{R}(\Sigma \setminus \{\gamma_0\})$ is of measure 0. For $y \in \Sigma \setminus [\mathcal{V}_\mathcal{R}(\Sigma \setminus \{\gamma_0\}) \cup S]$, $\mathcal{N}_\mathcal{R}(y) \cap (\Sigma \setminus \{\gamma_0\}) = \emptyset$ and hence $\mathcal{N}_\mathcal{R}(y) = \{\gamma_0\}$. Therefore, there exists $b > 0$ such that $\mathcal{R} = P(\nabla M_\Sigma(\gamma_0), b)$ on $\Sigma \setminus [\mathcal{V}_\mathcal{R}(\Sigma \setminus \{\gamma_0\}) \cup S]$. Since $\Sigma \setminus [\mathcal{V}_\mathcal{R}(\Sigma \setminus \{\gamma_0\}) \cup S]$ is dense on Σ, it gives rise to a contradiction.

To discuss the existence of solution in general case, we need to show the following L^∞ estimate.

Lemma 5.3. *Let $\mathcal{R} = \{\rho(x)x : x \in \Sigma\}$ be a weak solution in sense of (4-4) and $\inf_\Sigma \rho(x) = 1$. Assume that (4-3) holds and there exist ε, $\eta > 0$ such that for any $x \in \Sigma$*

$$\int_{N_\eta(x)} f\, dx \le \varepsilon,$$
$$\mu[N_\eta(x)] \le \mu(\Sigma) - 2\varepsilon,$$

where $N_\eta(x)$ denotes η-neighborhood of x on Σ. Then there exists $C > 0$ dependent on η, Σ but not ε such that

$$\sup_\Sigma \rho(x) \le C.$$

Proof. Assume $\rho(x_0) = \inf_{x \in \Sigma} \rho(x) = 1$. Let $P(m_0^*, b_0)$ be a supporting Σ-paraboloid to $\mathcal{R}$ at $\rho(x_0)x_0$. Obviously

$$1 = \frac{b_0}{1 - m_0^* x_0} = \rho(x_0) \le \rho(-\mathcal{D}(m_0^*)) \le \frac{b_0}{2}.$$

By Lemma 2.3, $x_0 = -\mathcal{D}(m_0^*)$, and hence $b_0 = 2$. We now claim

$$\mathcal{N}_{\mathcal{R}}[\Sigma \setminus N_\eta(\mathcal{D}(m_0^*))] \cap [\Sigma \setminus N_\eta(\mathcal{D}(m_0^*))] \neq \emptyset. \tag{5-6}$$

Otherwise, $\mathcal{N}_{\mathcal{R}}[\Sigma \setminus N_\eta(\mathcal{D}(m_0^*))] \subset N_\eta(\mathcal{D}(m_0^*))$. Hence $\Sigma \setminus N_\eta(\mathcal{D}(m_0^*)) \subset \mathcal{V}_{\mathcal{R}}[N_\eta(\mathcal{D}(m_0^*))]$ and one obtains the following contradiction

$$\begin{aligned}\int_\Sigma f\,dx - \varepsilon &\leq \int_{\Sigma \setminus N_\eta(\mathcal{D}(m_0^*))} f\,dx \\ &\leq \int_{\mathcal{V}_{\mathcal{R}}[N_\eta(\mathcal{D}(m_0^*))]} f\,dx = \mu(N_\eta(\mathcal{D}(m_0^*))) \leq \mu(\Sigma) - 2\varepsilon.\end{aligned}$$

By (5-6), there exist $D(m_1^*) \in \mathcal{N}_{\mathcal{R}}(y) \cap [\Sigma \setminus N_\eta(\mathcal{D}(m_0^*))]$ with $y \in \Sigma \setminus N_\eta(\mathcal{D}(m_0^*))$ and a supporting Σ-paraboloid $P(m_1^*, b_1)$ to $\mathcal{R}$ at $\rho(y)y$. Therefore

$$\frac{b_1}{1 - m_1^* y} = \rho(y) \leq \frac{2}{1 - m_0^* y}.$$

It follows from $y \notin N_\eta(\mathcal{D}(m_0^*))$ that $b_1 \leq C_\eta$ for some $c_\eta > 0$. Thus, we obtain

$$\rho(x) \leq \min\left\{\frac{2}{1 - m_0^* x}, \frac{b_1}{1 - m_1^* x}\right\}.$$

Since $|\mathcal{D}(m_1^*) - \mathcal{D}(m_0^*)| \geq \eta$ and $b_1 \leq C_\eta$, there exists C such that $\rho(x) \leq C$ on Σ. $\square$

Lemma 5.4. *Let μ be a Radon measure on Σ such that $\mu(\Sigma) > 0$ and μ is not a multiple of a Dirac measure. Then there exist ε_0, $\eta_0 > 0$ such that for any $\gamma \in \Sigma$*

$$\mu(N_{\eta_0}(\gamma)) \leq \mu(\Sigma) - 2\varepsilon_0. \tag{5-7}$$

Proof. Observe that there exists at most one $\gamma^* \in \Sigma$ such that $\mu(\{\gamma^*\}) > \frac{1}{2}\mu(\Sigma)$. We have $\mu(\{\gamma^*\}) < \mu(\Sigma)$. Otherwise, μ is concentrated on a point. Consequently, $\sup_{\gamma \in \Sigma} \mu(\{\gamma\}) < \mu(\Sigma)$. Choose $\varepsilon_0 > 0$ such that $\sup_{\gamma \in \Sigma} \mu(\{\gamma\}) \leq \mu(\Sigma) - 3\varepsilon_0$. To finish the proof, it suffices to show that there exists η_0 such that (5-7) holds. Otherwise, there exist $\eta_i \longrightarrow 0$ and $\gamma_i \in \Sigma \longrightarrow \gamma_0$ such that

$$\mu(N_{\eta_i}(\gamma_i)) > \mu(\Sigma) - 2\varepsilon_0.$$

Let $\varepsilon_i = |\gamma_i - \gamma_0|$. Obviously

$$\mu(N_{\eta_i + \varepsilon_i}(\gamma_0)) > \mu(\Sigma) - 2\varepsilon_0. \tag{5-8}$$

It is easy to verify that $\lim_{i \to \infty} N_{\eta_i + \varepsilon_i}(\gamma_0) = \{\gamma_0\}$. By taking limit in (5-8), we obtain $\mu(\{\gamma_0\}) \geq \mu(\Sigma) - 2\varepsilon_0$, which contradicts with the choice of ε_0. $\square$

We are now ready to establish the following main existence result.

Theorem 5.5. *Suppose that Σ is C^1 strictly convex. Let $f \in L^1(\Sigma)$ and $f \geq 0$. Let μ be a Radon measure on Σ that is not a multiple of a Dirac measure. Assume (4-3) holds. Then there exists a weak solution $\mathcal{R}$ to the Σ-reflector problem in sense of (4-4).*

Proof. By Lemma 5.4, there exist ε_0, $\eta_0 > 0$ such that (5-7) holds. For any integer $l \geq 1$, partition Σ into pairwise disjoint subsets $E_1^l, \cdots, E_{K_l}^l$ such that $\Sigma = \cup_{i=1}^{K_l} E_i^l$ and $diam(E_i^l) \leq \eta_1/l$ with $\eta_1 = \eta_0/2$. Choose $\gamma_i^l \in E_i^l$ and define $\mu_l = \sum_{i=1}^{K_l} \mu(E_i^l)\, \delta_{\gamma_i^l}$. We claim that

(i) $\mu_l \longrightarrow \mu$ weakly on Σ.
(ii) $\mu_l(\Sigma) = \mu(\Sigma)$ for $l \geq 1$.
(iii) $\mu_l(N_{\eta_1}(\gamma)) \leq \mu_l(\Sigma) - 2\varepsilon_0$, for $\gamma \in \Sigma$, $l \geq 1$.

It is easy to verify (ii). (i) follows from the following observation that for $h \in C(\Sigma)$

$$\int_\Sigma h(x)\, d\mu_l - \int_\Sigma h(x)\, d\mu = \sum_{i=1}^{K_l} \int_{E_i^l} [h(\gamma_i^l) - h(x)]\, d\mu = o(1)\mu(\Sigma).$$

To prove (iii), for $\gamma \in \Sigma$

$$\mu_l(N_{\eta_1}(\gamma)) = \sum_{\gamma_j^l \in N_{\eta_1}(\gamma)} \mu(E_j^l) = \mu\left(\cup_{\gamma_j^l \in N_{\eta_1}(\gamma)} E_j^l\right).$$

If $\gamma_j^l \in N_{\eta_1}(\gamma)$, then $E_j^l \subset N_{2\eta_1}(\gamma)$. Therefore, $\mu_l(N_{\eta_1}(\gamma)) \leq \mu((N_{\eta_0}(\gamma))$ and (iii) is implied.

By Theorem 5.1, there exists a weak solution $\mathcal{R}_l$ parameterized by $\rho_l(x)x$ satisfying

$$\mu_{f,\mathcal{R}_l} = \mu_l. \tag{5-9}$$

We may normalize $\mathcal{R}_l$ such that $\inf_\Sigma \rho_l(x) = 1$. By Lemma 5.3, there exists a constant C independent of l such that $\sup_\Sigma \rho_l(x) \leq C$ for $l \geq 1$.

Now estimate the Lipschitz constant of ρ_l. Given $x_0 \in \Sigma$, let $P(m_0^*, b_0)$ be a supporting Σ-paraboloid to $\mathcal{R}_l$ at $\rho_l(x_0)x_0$. Therefore, $\rho_l(x)(1 - m_0^* x)$ attains the maximum b_0 at x_0. Since $1 \leq \rho_l \leq C$, $2 \leq b_0 \leq C$. For $x_1 \in \Sigma$ close to x_0, we have

$$\rho_l(x_1) - \rho_l(x_0) \leq \frac{b_0 m_0^*(x_1 - x_0)}{(1 - m_0^* x_1)(1 - m_0^* x_0)} \leq C|x_1 - x_0|,$$

which gives an estimate of the Lipschitz constant of ρ_l.

By the Arzelà-Ascoli theorem, we may assume that $\rho_l \longrightarrow \rho$ uniformly on Σ. Clearly, $\mathcal{R} = \{\rho(x)x : x \in \Sigma\}$ is a Σ-reflector. To finish the proof, we want to show that $\mathcal{R}$ is a weak solution to the Σ-reflector problem. If we can show that $\mu_{f,\mathcal{R}_l}$ weakly converges to $\mu_{f,\mathcal{R}}$, by taking limit in (5-9), we obtain $\mu_{f,\mathcal{R}} = \mu$, i.e, $\mathcal{R}$ is a weak solution. To show that $\mu_{f,\mathcal{R}_l} \longrightarrow \mu_{f,\mathcal{R}}$ weakly, note by Lemma 4.6 that for any compact set $K \subset \Sigma$

$$\varlimsup_{l\to\infty} \mu_{f,\mathcal{R}_l}(K) \leq \int_{\varlimsup_{l\to\infty} \mathcal{V}_{\mathcal{R}_l}(K)} f\, dx \leq \int_{\mathcal{V}_{\mathcal{R}}(K)} f\, dx = \mu_{f,\mathcal{R}}(K),$$

and similarly for any open set $G \subset \Sigma$, $\varliminf_{l\to\infty} \mu_{f,\mathcal{R}_l}(G) \geq \mu_{f,\mathcal{R}}(G)$. $\square$

In light of Remark 5.2, Theorem 5.5 may fail if μ is a Dirac measure.

§6. Uniqueness of Weak Solutions

The purpose in this section is to establish the uniqueness of weak solutions to the Σ-reflector problem up to dilations.

If $P(m^*, b_0)$ is a Σ-paraboloid intersecting with a Σ-reflector $\mathcal{R}$, define

$$\mathcal{S}_{\mathcal{R}}(P(m^*, b_0)) = \{X \in \mathcal{R} : \exists\, b \geq b_0 \text{ such that } X \in P(m^*, b)\}.$$

We first give two auxiliary lemmas.

Lemma 6.1. *Let $\mathcal{R}$ be a Σ-reflector and $P(m_0^*, b_0)$ be a Σ-paraboloid such that $\mathcal{R} \cap P(m_0^*, b_0) \neq \emptyset$. Then for X_1, $X_2 \in \mathcal{S}_{\mathcal{R}}(P(m_0^*, b_0))$, there exists a planar curve on $\mathcal{S}_{\mathcal{R}}(P(m_0^*, b_0))$ joining X_1 to X_2.*

Proof. Let X_1, X_2 be two distinct points on $\mathcal{S}_{\mathcal{R}}(P(m_0^*, b_0))$. First consider the case that $X_2 - X_1 = t\mathcal{D}(m_0^*)$ for some $-\infty < t < \infty$. Let L be the two-dimensional plane generated by O, X_1, and X_2, and ℓ be the line passing X_1, X_2 and with the direction $\mathcal{D}(m_0^*)$. Let T_{Y_0} be the tangent hyperplane to $P(m_0^*, b_0)$ at the point Y_0 where ℓ intersects $P(m_0^*, b_0)$. Obviously, the line $T_{Y_0} \cap L$ traverses the closed convex curve $\mathcal{R} \cap L$ and X_1, X_2 are below $T_{Y_0} \cap L$ in the direction $\mathcal{D}(m_0^*)$, i.e., $Y_0 - X_i = \lambda_i \mathcal{D}(m_0^*)$ with $\lambda_i > 0$, for $i = 1, 2$. Therefore, there exists a piece of $\mathcal{R} \cap L$ denoted by $\mathcal{C}$ which connects X_1 and X_2 and lies below $T_{Y_0} \cap L$ in the direction of $\mathcal{D}(m_0^*)$. Clearly, $\mathcal{C}$ lies outside $P(m_0^*, b_0)$.

It remains to prove the lemma for the case that $X_2 - X_1$ is not parallel to $\mathcal{D}(m_0^*)$. Let $\widetilde{L}$ be the two-dimensional plane generated by X_1 and the line passing through X_2 with the direction $\mathcal{D}(m_0^*)$. Denote by $\widetilde{\mathcal{C}}$ the portion of the convex curve $\mathcal{R} \cap \widetilde{L}$ connecting X_1 and X_2 which lies below the segment $\overline{X_1 X_2}$ in the direction $\mathcal{D}(m_0^*)$. It is convenient to view geometrically the direction of $\mathcal{D}(m_0^*)$ as the positive y-axis direction in the plane $\widetilde{L}$ in the proof. We claim that $\widetilde{\mathcal{C}}$ lies outside $P(m_0^*, b_0)$. Indeed, let $X \in \widetilde{\mathcal{C}}$ and $P(m^*, b)$ be a supporting Σ-paraboloid to $\mathcal{R}$ at the point X. If $m^* = m_0^*$, then $b_0 \leq b$ and X is outside $P(m_0^*, b_0)$. Now consider the case $m^* \neq m_0^*$. Since X_1, X_2 are outside $P(m_0^*, b_0)$ and inside $P(m^*, b)$, $m_0^* X_i + b_0 \leq m^* X_i + b$, for $i = 1, 2$. Because $(m_0^* - m^*)\mathcal{D}(m_0^*) = 1 - m^*\mathcal{D}(m_0^*) > 0$, X_1, X_2 lie below the line $\Pi \cap \widetilde{L}$ in the direction $\mathcal{D}(m_0^*)$, where Π is the hyperplane with equation $m_0^* Y + b_0 = m^* Y + b$. It is easy to see that either the closed convex curve $\mathcal{R} \cap \widetilde{L}$ lies below the line $\Pi \cap \widetilde{L}$ in the direction $\mathcal{D}(m_0^*)$ or $\Pi \cap \widetilde{L}$ crosses $\mathcal{R} \cap \widetilde{L}$. In either case, $\widetilde{\mathcal{C}}$ lies below $\Pi \cap \widetilde{L}$ in the direction $\mathcal{D}(m_0^*)$. Therefore, $m_0^* X + b_0 \leq m^* X + b$ and X is outside $P(m_0^*, b_0)$. $\square$

Lemma 6.2. *Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two Σ-reflectors parameterized by $\rho_1(x)$, $\rho_2(x)$, respectively. Suppose that for a.e. $x \in \Sigma$, the unit outward normal of $\mathcal{R}_1$ at $\rho_1(x)x$ is the same as the unit outward normal of $\mathcal{R}_2$ at $\rho_2(x)x$. Then $\rho_1 = C\rho_2$ on Σ for some constant C.*

Proof. Given $x_0 \in \Sigma$, without loss of generality, choose a local coordinate system such that some neighborhood $U \subset \Sigma$ of x_0 can be written as $U = \{x(x') = (x', u(x')) : x' \in B_r(x_0')\}$, where $u(x')$ is a C^1 function defined in $B_r(x_0')$, $B_r(x_0')$ is the $(n-1)$-dimensional Euclidean ball centered at x_0' with radius r, and $x_0 = (x_0', u(x_0'))$. For $i = 1, 2$, $\bar{\rho}_i(x') = \rho_i(x(x'))$ is a Lipschitz function on $B_r(x_0')$. Consider a point $y' \in B_r(x_0')$ such that y' is a differentiable point of both $\bar\rho_1$ and $\bar\rho_2$, and $\mathcal{R}_1$, $\mathcal{R}_2$ have the unit outward normal $\nu(y')$ at $\bar\rho_1(y')x(y')$, $\bar\rho_2(y')x(y')$, respectively. For $1 \le k \le n-1$, let $\beta_k(t) = y' + te_k$, where $|t| < \varepsilon$ for some small ε and e_k is the k-th unit vector of the standard Euclidean orthogonal basis. Thus, $x(\beta_k(t))$ is a curve on Σ, and $\bar\rho_i(\beta_k(t))\, x(\beta_k(t))$ is a curve on $\mathcal{R}_i$ passing through $\bar\rho_i(y')x(y')$ for $i = 1, 2$. We have

$$\nu(y')\left(\frac{d}{dt}x(\beta_k(t))\bigg|_{t=0} + x(y')\frac{\nabla_k\bar\rho_i(y')}{\bar\rho_i(y')}\right) = 0, \qquad i = 1,\ 2.$$

By subtracting them, we obtain $\nabla \log\bar\rho_1(y') = \nabla\log\bar\rho_2(y')$, which yields $\rho_1 = C\rho_2$ on U. By the connectedness of Σ, $\rho_1 = C\rho_2$ on Σ. $\square$

Below is our main uniqueness theorem.

Theorem 6.3. *Suppose that Σ is strictly convex and C^1. Let $f \in L^1(\Sigma)$ and $f \ge 0$. Let μ be a Radon measure on Σ but not a multiple of a Dirac measure. Assume $\int_G f\,dx > 0$ for any open set $G \subset \Sigma$. If $\mathcal{R}_1 = \{\rho_1(x)x : x \in \Sigma\}$ and $\mathcal{R}_2 = \{\rho_2(x)x : x \in \Sigma\}$ are two weak solutions to the Σ-reflector problem in sense of (4-4), then $\rho_1 = C\rho_2$ on Σ for some constant C.*

Proof. Let $S \subset \Sigma$ be the set of singular points of $\mathcal{R}_1$ and $\mathcal{R}_2$. Then S has the area measure 0. For $x \in \Sigma\setminus S$, $i = 1, 2$, $\mathcal{N}_{\mathcal{R}_i}(x)$ has only one element and for simplicity we will make no difference about $\mathcal{N}_{\mathcal{R}_i}(x)$ and its element. If we can show $\mathcal{N}_{\mathcal{R}_1}(x) = \mathcal{N}_{\mathcal{R}_2}(x)$ for $x \in \Sigma\setminus S$, then $\mathcal{R}_1$ and $\mathcal{R}_2$ have the same unit outward normal at $\rho_1(x)x$ and $\rho_2(x)x$ respectively. By Lemma 6.2, there exists C such that $\rho_1 = C\rho_2$. We proceed by contradiction. Assume that $\mathcal{N}_{\mathcal{R}_1}(x_0) \ne \mathcal{N}_{\mathcal{R}_2}(x_0)$ for some $x_0 \in \Sigma\setminus S$. Through a dilation, one can assume $\rho_1(x_0) = \rho_2(x_0)$. Since $\mathcal{N}_{\mathcal{R}_1}(x_0) \ne \mathcal{N}_{\mathcal{R}_2}(x_0)$, $\mathcal{R}_1$ cannot be contained inside $\mathcal{R}_2$ and $\Omega_0 = \{x \in \Sigma : \rho_1 > \rho_2\} \ne \emptyset$. Moreover, the tangent plane of $\mathcal{R}_2$ at $\rho_2(x_0)x_0$ cannot be the tangent plane of $\mathcal{R}_1$ at $\rho_1(x_0)x_0$ and must cross $\mathcal{R}_1$. Hence, x_0 is a boundary point of Ω_0. Let Ω be the connected component of Ω_0 such that $x_0 \in \partial\Omega$. We first prove four claims.

Claim 1: Let $\gamma_0 = \mathcal{D}(m_0^*) = \mathcal{N}_{\mathcal{R}_1}(x_0)$. Then $\gamma_0 \notin \mathcal{N}_{\mathcal{R}_2}(\overline{\Omega})$.

To prove, assume otherwise $\gamma_0 \in \mathcal{N}_{\mathcal{R}_2}(y_0)$ where $y_0 \in \overline{\Omega} \subset \{x \in \Sigma : \rho_1 \ge \rho_2\}$. Let $P(m_0^*, b_2)$ be the supporting Σ-paraboloid to $\mathcal{R}_2$ at $\rho_2(y_0)y_0$ and $P(m_0^*, b_1)$ be the supporting Σ-paraboloid to $\mathcal{R}_1$ at $\rho_1(x_0)x_0$. If $b_2 = b_1$, then $\rho_2(x_0) = \rho_1(x_0) = b_2/(1 - m_0^*x_0)$. Thus, $\gamma_0 \in \mathcal{N}_{\mathcal{R}_2}(x_0)$ and that is impossible. If $b_2 > b_1$, we get the contradiction

$$\rho_2(y_0) = b_2/(1 - m_0^*y_0) > b_1/(1 - m_0^*y_0) \ge \rho_1(y_0),$$

and if $b_2 < b_1$, we get another contradiction

$$\rho_1(x_0) = b_1/(1 - m_0^*x_0) > b_2/(1 - m_0^*x_0) \ge \rho_2(x_0).$$

Claim 2: $\mathcal{N}_{\mathcal{R}_1}(x)$ is continuous at x_0. More precisely, for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$|\gamma - \gamma_0| < \varepsilon, \qquad \text{for } \gamma \in \mathcal{N}_{\mathcal{R}_1}(x) \text{ with } |x - x_0| < \eta.$$

Claim 2 follows directly from the fact that $\mathcal{N}_{\mathcal{R}_1}(x_0)$ is single-valued.

Claim 3: $\Omega^* = \mathcal{N}_{\mathcal{R}_2}(\Omega) \subset \mathcal{N}_{\mathcal{R}_1}(\Omega)$.

The proof is to use dragging Σ-paraboloid as in the proof of [CGH05, Theorem 3.4]. Let $\mathcal{D}(m^*) \in \Omega^*$ and $P(m^*, b)$ be a supporting Σ-paraboloid of $\mathcal{R}_2$ at $\rho_2(y)y$ with $y \in \Omega$. Geometrically, $P(m^*, b)$ traverses the portion $\mathcal{R}_1|_\Omega = \{\rho_1(x)x : x \in \Omega\}$ of $\mathcal{R}_1$. Drag $P(m^*, b)$ outward (i.e., have b increase) and we obtain a Σ-paraboloid $P(m^*, b_0)$ with $b_0 > b$ supporting $\mathcal{R}_1|_\Omega$ at some point $\rho_1(z_0)z_0$ with $z_0 \in \Omega$. By Lemma 6.1, $\mathcal{R}_1|_{\Sigma\setminus\Omega}$ is also contained inside $P(m^*, b_0)$. Otherwise, $P(m^*, b_0)$ must touches $\mathcal{R}_1$ at some $\rho_1(z_1)z_1$ where $z_1 \in \partial\Omega$. Since $b_0 > b$, $\rho_1(z_1) > b/(1 - m^*z_1) \geq \rho_2(z_1)$, which contradicts with $z_1 \in \partial\Omega$. Therefore, $P(m^*, b_0)$ supports $\mathcal{R}_1$ at $\rho_1(z_0)z_0$ and $\mathcal{D}(m^*) \in \mathcal{N}_{\mathcal{R}_1}(\Omega)$.

Claim 4: $\Omega \subset \mathcal{V}_{\mathcal{R}_2}(\Omega^*)$ and $\mathcal{V}_{\mathcal{R}_1}(\Omega^*) \subset \Omega \setminus N_\eta(x_0)$ for some $\eta > 0$.

Obviously, $\mathcal{V}_{\mathcal{R}_2}(\Omega^*) = \{y \in \Sigma : \mathcal{N}_{\mathcal{R}_2}(y) \cap \Omega^* \neq \emptyset\} \supset \Omega$. Now prove $\mathcal{V}_{\mathcal{R}_1}(\Omega^*) \subset \Omega \setminus N_\eta(x_0)$. Given $\gamma = \mathcal{D}(m^*) \in \Omega^*$ and $y \in \mathcal{V}_{\mathcal{R}_1}(\gamma)$, let $P(m^*, \bar{b}_1)$ be the supporting Σ-paraboloid to $\mathcal{R}_1$ at $\rho_1(y)y$. By claim 3, $\gamma \in \mathcal{N}_{\mathcal{R}_1}(\Omega)$ and $P(m^*, \bar{b}_1)$ must touch $\mathcal{R}_1$ at some $\rho_1(\bar{z}_1)\bar{z}_1$ where $\bar{z}_1 \in \Omega$. We assert $y \in \Omega$. Otherwise, by Lemma 6.1, $P(m^*, \bar{b}_1)$ must touch $\mathcal{R}_1$ at some $\rho_1(\bar{z})\bar{z}$ where $\bar{z} \in \partial\Omega$. Since $\gamma \in \mathcal{N}_{\mathcal{R}_2}(\Omega)$ and $\rho_2 < \rho_1$ in Ω, there exists $\bar{b}_2 < \bar{b}_1$ such that $P(m^*, \bar{b}_2)$ supports $\mathcal{R}_2$. Thus, we obtain $\rho_1(\bar{z}) > \bar{b}_2/(1 - m^*\bar{z}) \geq \rho_2(\bar{z})$ which contradicts with $\bar{z} \in \partial\Omega$. Thus, we have shown $\mathcal{V}_{\mathcal{R}_1}(\Omega^*) \subset \Omega$. To show $\mathcal{V}_{\mathcal{R}_1}(\Omega^*) \cap N_\eta(x_0) = \emptyset$, note that $\mathcal{N}_{\mathcal{R}_2}(\overline{\Omega})$ is a closed set containing Ω^*, and by claim 1 and claim 2, $\mathcal{N}_{\mathcal{R}_1}(x) \cap \Omega^* = \emptyset$ for $x \in N_\eta(x_0)$ where η is small enough. Therefore, $\mathcal{N}_{\mathcal{R}_1}(N_\eta(x_0)) \cap \Omega^* = \emptyset$ and $\mathcal{V}_{\mathcal{R}_1}(\Omega^*) \subset \Omega \setminus N_\eta(x_0)$.

To finish the proof of the theorem, observe that Ω^* is a F_σ set. Since $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfy (4-4), we have

$$\int_{\mathcal{V}_{\mathcal{R}_1}(\Omega^*)} f\,dx = \mu(\Omega^*) = \int_{\mathcal{V}_{\mathcal{R}_2}(\Omega^*)} f\,dx.$$

By claim 4, it implies $\int_{\Omega\cap N_\eta(x_0)} f\,dx \leq 0$ and that is a contradiction. $\square$

We point out that the condition $\int_G f\,dx > 0$ for any open set $G \subset \Sigma$ in Theorem 6.3 is slightly weaker than $f > 0$ a.e. on Σ, in light of sets of Cantor type. This condition is also sharp to obtain the uniqueness result. Indeed, if some region on a Σ-reflector $\mathcal{R}$ is not illuminated, one can modify that piece of $\mathcal{R}$, and the uniqueness is violated. See the following example.

Example 6.4. Let $\Sigma = S^{n-1}$. Let $\mathcal{R}_1 = S^{n-1}$ be parameterized by $\rho_1(x) \equiv 1$ and $\mathcal{R}_2$ parameterized by $\rho_2(x) = \min\left\{1, \dfrac{2(1-\varepsilon)}{1+e_n x}\right\}$ where $0 < \varepsilon < 1$ and $e_n = (0, \cdots, 0, 1)$. Consider $f(x) = \chi_\Omega(x)$, the characteristic function of $\Omega = \{x \in S^{n-1} : 1 < \frac{2(1-\varepsilon)}{1+e_n x}\}$, and the Radon measure μ defined by

$$\mu(E) = \int_{\mathcal{V}_{\mathcal{R}_1}(E)} f\,dx = area[(-E) \cap \Omega],$$

where E is any Borel set on S^{n-1} and $-E = \{-x : x \in E\}$. Then $\mathcal{R}_1$ and $\mathcal{R}_2$ are two non-homothetic weak solutions to the reflector problem satisfying (4-4) and the uniqueness fails.

Proof. Obviously, $\mathcal{N}_{\mathcal{R}_1}(x) = -x$ for $x \in S^{n-1}$, $\mathcal{V}_{\mathcal{R}_1}(\gamma) = -\gamma$ for $\gamma \in S^{n-1}$, and $\mathcal{R}_1$ is a reflector. It is also easy to verify that $\mathcal{N}_{\mathcal{R}_2}(x) = -x$ for $x \in \Omega$, $\mathcal{N}_{\mathcal{R}_2}(\Omega_1) = \{-e_n\}$ where $\Omega_1 = \{x \in S^{n-1} : \frac{2(1-\varepsilon)}{1+e_n x} < 1\}$, $\mathcal{N}_{\mathcal{R}_2}(\partial\Omega) = -\overline{\Omega_1}$, and $\mathcal{R}_2$ is a reflector consisting of pieces of S^{n-1} and $P(-e_n, 2(1-\varepsilon))$. We claim that for any Borel set $E \subset S^{n-1}$

$$[\mathcal{V}_{\mathcal{R}_1}(E) \setminus \mathcal{V}_{\mathcal{R}_2}(E)] \cup [\mathcal{V}_{\mathcal{R}_2}(E) \setminus \mathcal{V}_{\mathcal{R}_1}(E)] \subset \Omega^c \tag{6-1}$$

Indeed, if $x \in \mathcal{V}_{\mathcal{R}_1}(E) \cap \Omega$, then $-x = \mathcal{N}_{\mathcal{R}_1}(x) = \mathcal{N}_{\mathcal{R}_2}(x) \in E$. Thus, $x \in \mathcal{V}_{\mathcal{R}_2}(E)$ and $\mathcal{V}_{\mathcal{R}_1}(E) \setminus \mathcal{V}_{\mathcal{R}_2}(E) \subset \Omega^c$. Similarly, $\mathcal{V}_{\mathcal{R}_2}(E) \setminus \mathcal{V}_{\mathcal{R}_1}(E) \subset \Omega^c$. This proves (6-1). Therefore, by (6-1), for any Borel set $E \subset S^{n-1}$

$$\mu(E) = \int_{\mathcal{V}_{\mathcal{R}_1}(E)} f\, dx = \int_{\mathcal{V}_{\mathcal{R}_2}(E)} f\, dx,$$

and $\mathcal{R}_1$, $\mathcal{R}_2$ are two non-homothetic weak solutions. □

Corollary 6.5. *Suppose that Σ is strongly convex. Let f, $g \in L^1(\Sigma)$ and f, $g \geq 0$, and let $d\mu = g\, d\gamma$. Assume $\int_G g\, d\gamma > 0$ for any open set $G \subset \Sigma$. If $\mathcal{R}_1 = \{\rho_1(x)x : x \in \Sigma\}$ and $\mathcal{R}_2 = \{\rho_2(x)x : x \in \Sigma\}$ are two weak solutions to the Σ-reflector problem satisfying (4-4), then $\rho_1 = C\rho_2$ on Σ for some constant C.*

Proof. We first show that for any Σ-reflector $\mathcal{R}$ and any Borel set $E^* \subset \Sigma^*$

$$\mathcal{F}_{\mathcal{R}}(\mathcal{D}(E^*)) = \mathcal{V}_{\mathcal{R}^*}(E^*). \tag{6-2}$$

To prove (6-2), let $\beta_0^* \in \mathcal{F}_{\mathcal{R}}(\mathcal{D}(m_0^*))$ where $m_0^* \in E^*$. Then $\mathcal{D}(m_0^*) \in \mathcal{F}_{\mathcal{R}^*}(\beta_0^*)$ and $m_0^* \in \nabla M_\Sigma \mathcal{F}_{\mathcal{R}^*}(\beta_0^*) = \mathcal{N}_{\mathcal{R}^*}(\beta_0^*)$. So $\beta_0^* \in \mathcal{V}_{\mathcal{R}^*}(m_0^*)$. To establish the reverse inclusion, given $\beta_1^* \in \mathcal{V}_{\mathcal{R}^*}(m_1^*)$ with $m_1^* \in E^*$, we have $m_1^* \in \mathcal{N}_{\mathcal{R}^*}(\beta_1^*)$ and $\nabla M_{\Sigma^*}(m_1^*) \in \mathcal{F}_{\mathcal{R}^*}(\beta_1^*)$. Therefore, $\beta_1^* \in \mathcal{F}_{\mathcal{R}^{**}}(\mathcal{D}(m_1^*)) = \mathcal{F}_{\mathcal{R}}(\mathcal{D}(m_1^*))$. The proof of (6-2) is done.

Let $f^*(\beta^*) = f(\mathcal{D}(\beta^*))J_{\mathcal{D}}(\beta^*)$ and $g^*(m^*) = g(\mathcal{D}(m^*))J_{\mathcal{D}}(m^*)$. By change of variables, Lemma 4.12, and (6-2), if $\mathcal{R}$ is a weak solution in the sense of (4-4), then we have for Borel set $E^* \subset \Sigma^*$

$$\int_{E^*} f^*(\beta^*)\, d\beta^* = \int_{\mathcal{D}(E^*)} f(x)\, dx = \int_{\mathcal{F}_{\mathcal{R}}[\mathcal{D}(E^*)]} g^*(m^*)\, dm^* = \int_{\mathcal{V}_{\mathcal{R}^*}(E^*)} g^*\, dm^*,$$

which simply says that $\mathcal{R}^*$ is a weak solution to the Σ^*-reflector problem with emitted illumination g^* and reflected illumination f^*. Therefore, by Theorem 6.3, if $\mathcal{R}_1$ and $\mathcal{R}_2$ are two weak solutions satisfying (4-4), then $\mathcal{R}_1^* = C\mathcal{R}_2^*$ for some $C > 0$. By Lemma 4.10, $\mathcal{R}_1 = \frac{1}{C}\mathcal{R}_2$. □

We mention that in the case $\Sigma = S^{n-1}$, Corollary 6.5 is due to [GW98].

§7. The Monge-Kantorovich Mass Transfer Approach

In this section, we use a variational method to re-establish the existence of solutions of the global Σ-reflector problem via its connection with the optimal mass transport problem of the cost function $c(x, m^*) = -\log(1 - xm^*)$ defined on $\Sigma \times \Sigma^*$ which is singular at all points $(x, \nabla M_\Sigma(x))$.

Throughout this section, assume that Σ is C^2 strongly convex, and consider the global Σ-reflector problem with emitted illumination intensity $f \in L^1(\Sigma)$ and prescribed reflected illumination intensity $d\mu = g d\gamma$ with $g \in L^1(\Sigma)$ such that (4-3) holds.

Let $\mathcal{R}$ parameterized by $\rho(x)x$ for $x \in \Sigma$ be a weak solution satisfying (4-4). By the definition of the Legendre transform, one can verify

$$\log \rho(x) + \log \rho^*(m^*) = c(x, m^*), \qquad \text{for } x \in \Sigma,\ m^* \in \mathcal{F}_\mathcal{R}(x). \tag{7-1}$$

Note that $\mathcal{F}_\mathcal{R}$ is a single-valued mapping on $\Sigma \setminus S_\mathcal{R}$, where $S_\mathcal{R}$ is the singular set of $\mathcal{R}$ of measure zero. Without loss of generality, we can assume that $S_\mathcal{R}$ is a Borel set.

Denote by $C(\Sigma)$ ($C(\Sigma^*)$) the space of all continuous functions on Σ (Σ^*).

Lemma 7.1. *Let $\mathcal{R}$ be a Σ-reflector. Then $\mathcal{R}$ is a weak solution satisfying (4-4) if and only if for any $v^* \in C(\Sigma^*)$*

$$\int_\Sigma v^*(\mathcal{F}_\mathcal{R}(x)) f(x)\, dx = \int_{\Sigma^*} v^*(m^*) g^*(m^*)\, dm^*, \tag{7-2}$$

where g^* is defined as in Lemma 4.12.

Proof. Let $\mathcal{R}$ be a weak solution. To show (7-2), it suffices to prove it for $v^* = \chi_{E^*}$, the characteristic function of the Borel set E^*. Note that $m_0^* \in \mathcal{F}_\mathcal{R}(x_0)$ iff $x_0 \in \mathcal{F}_{\mathcal{R}^*}(m_0^*)$. It is easy to verify $\mathcal{F}_\mathcal{R}[(\Sigma \setminus S_\mathcal{R}) \cap \mathcal{F}_{\mathcal{R}^*}(E^*)] \subset E^*$ and $\mathcal{F}_\mathcal{R}[\mathcal{F}_{\mathcal{R}^*}(E^*)] \supset E^*$. We can assume that $\mathcal{F}_{\mathcal{R}^*}(E^*)$ is a Borel set. Otherwise, replace it by a suitable Borel set with the same measure, as in the proof of Lemma 4.12. Therefore, by Lemma 4.12, we obtain

$$\begin{aligned}\int_{E^*} g^*\, dm^* \le \int_{\mathcal{F}_{\mathcal{R}^*}(E^*)} f\, dx &= \int_\Sigma \chi_{E^*}(\mathcal{F}_\mathcal{R}(x)) f(x)\, dx \\ &= \int_{(\Sigma \setminus S_\mathcal{R}) \cap \mathcal{F}_{\mathcal{R}^*}(E^*)} f(x)\, dx \le \int_{E^*} g^*\, dm^*.\end{aligned}$$

To prove sufficient condition, assume that (7-2) holds. We now show for any open set $G^* \subset \Sigma^*$

$$\int_{\mathcal{F}_{\mathcal{R}^*}(G^*)} f\, dx \le \int_{G^*} g^*\, dm^*. \tag{7-3}$$

Indeed, given a compact set $K^* \subset G^*$, choose $v^* \in C(\Sigma^*)$ such that $0 \le v^* \le 1$, $v^* = 1$ on K^*, and $v^* = 0$ outside G^*. Since $\mathcal{F}_\mathcal{R}[(\Sigma \setminus S_\mathcal{R}) \cap \mathcal{F}_{\mathcal{R}^*}(K^*)] \subset K^*$, we have

$$\int_{(\Sigma \setminus S_\mathcal{R}) \cap \mathcal{F}_{\mathcal{R}^*}(K^*)} f(x)\, dx \le \int_\Sigma v^*(\mathcal{F}_\mathcal{R}(x)) f(x)\, dx \le \int_{G^*} g^*\, dm^*.$$

(7-3) follows from arbitrariness of K^*. Because a Borel set can be approximated by open sets, (7-3) is still valid for Borel sets in Σ^*. Obviously, $\mathcal{F}_{\mathcal{R}^*}[\nabla M_\Sigma(E)] = \mathcal{V}_{\mathcal{R}}(E)$ for $E \subset \Sigma$. From (7-3) and by change of variables, we have for Borel set $E \subset \Sigma$

$$\int_{\mathcal{V}_{\mathcal{R}}(E)} f\, dx \le \int_E g\, d\gamma.$$

Noting (4-1) and (4-3) which is simply (7-2) with $v^* = 1$, we apply the above inequality to $\Sigma \setminus E$ to get the reverse inequality and (4-4). $\square$

Now turn to the variational formulation of the Σ-reflector problem. Consider the set of admissible functions

$$\mathcal{K} = \{(u, v^*) \in C(\Sigma) \times C(\Sigma^*) : \ u(x) + v^*(m^*) \le c(x, m^*) \text{ on } \Sigma \times \Sigma^*\}$$

where $c(x, \nabla M_\Sigma(x))$ is understood as ∞ and the dual functional

$$I(u, v^*) = \int_\Sigma u(x) f(x)\, dx + \int_{\Sigma^*} v^*(m^*) g^*(m^*)\, dm^*, \qquad \text{for } u \in C(\Sigma),\ v \in C(\Sigma^*).$$

Given $\varphi \in C(\Sigma)$, $\psi^* \in C(\Sigma^*)$, let $\varphi^c = (\varphi)^c$ and $\psi^*_c = (\psi^*)_c$ denote the c-transform of φ and c^*-transform of ψ^* respectively defined by

$$\varphi^c(m^*) = \inf_{x \in \Sigma} \left[c(x, m^*) - \varphi(x)\right],$$
$$\psi^*_c(x) = \inf_{m^* \in \Sigma^*} \left[c(x, m^*) - \psi^*(m^*)\right].$$

It is easy to verify that $\mathcal{R} = \{e^{\psi^*_c(x)} x : x \in \Sigma\}$ is a Σ-reflector and $\widehat{\mathcal{R}} = \{e^{\varphi^c(m^*)} m^* : m^* \in \Sigma^*\}$ is a Σ^*-reflector.

Lemma 7.2. *Suppose that Σ is strongly convex. Then*

(i) *If $\mathcal{R}$ is a weak solution of the Σ-reflector problem satisfying (4-4), then* $(\log \rho, \log \rho^*)$ *is a maximizer of the functional $I(u, v^*)$ in $\mathcal{K}$, where ρ and ρ^* are radial functions of $\mathcal{R}$ and $\mathcal{R}^*$, respectively.*

(ii) *If (φ, ψ^*) maximizes $I(u, v^*)$ in $\mathcal{K}$, then $\mathcal{R}_0 = \{e^{\varphi_0(x)} x : x \in \Sigma\}$ is a weak solution satisfying (4-4), where $\varphi_0 = \psi^*_c$.*

Proof. First prove (i). Given $(u, v^*) \in \mathcal{K}$, then by (7-1)

$$u(x) + v^*(\mathcal{F}_{\mathcal{R}}(x)) \le c(x, \mathcal{F}_{\mathcal{R}}(x)) = \log \rho(x) + \log \rho^*(\mathcal{F}_{\mathcal{R}}(x)), \qquad \text{a.e. } x \text{ in } \Sigma.$$

Integrating the above inequality with respect to $f dx$ yields

$$\int_\Sigma u f\, dx + \int_\Sigma v^*(\mathcal{F}_{\mathcal{R}}(x)) f(x)\, dx \le \int_\Sigma \log \rho f\, dx + \int_\Sigma \log \rho^*(\mathcal{F}_{\mathcal{R}}(x)) f(x)\, dx.$$

By Lemma 7.1, it implies $I(u, v^*) \leq I(\log\rho, \log\rho^*)$. Clearly, $(\log\rho, \log\rho^*) \in \mathcal{K}$ by the definition of Legendre transform. Thus, $I(\log\rho, \log\rho^*) = \sup_{(u,v^*)\in\mathcal{K}} I(u, v^*)$.

To prove (ii), first observe that $(\varphi_0^c)_c = \varphi_0$, where $\varphi_0^c = (\varphi_0)^c$. Indeed, $\mathcal{R}_0$ is a Σ-reflector, and $e^{\varphi_0^c(m^*)}$ and $e^{(\varphi_0^c)_c(x)}$ are the radial functions of the Legendre transform $\mathcal{R}_0^*$ and double Legendre transform $\mathcal{R}_0^{**}$ of $\mathcal{R}_0$, respectively. By Lemma 4.10, $(\varphi_0^c)_c = \varphi_0$. For simplicity, set $\psi_0^* = \varphi_0^c$ and then $\varphi_0 = (\psi_0^*)_c$. Obviously, $\varphi \leq \varphi_0$ and $\psi^* \leq \psi_0^*$. Hence, (φ_0, ψ_0^*) also maximizes $I(u, v^*)$ in $\mathcal{K}$.

We now show that $\mathcal{R}_0$ satisfies (7-2) to finish the proof. Given $v^* \in C(\Sigma^*)$ and a small number θ, consider $\psi_\theta^* = \psi_0^* + \theta v^*$ and $\varphi_\theta(x) = \inf_{m^*\in\Sigma^*}[c(x, m^*) - \psi_\theta^*(m^*)]$. Clearly, $(\varphi_\theta, \psi_\theta^*) \in \mathcal{K}$. For $\bar{x} \in \Sigma$, let $\varphi_0(\bar{x}) = c(\bar{x}, \overline{m}^*) - \psi_0^*(\overline{m}^*)$ and $\varphi_\theta(\bar{x}) = c(\bar{x}, \overline{m}_\theta^*) - \psi_\theta^*(\overline{m}_\theta^*)$. Obviously

$$-\theta v^*(\overline{m}_\theta^*) \leq \varphi_\theta(\bar{x}) - \varphi_0(\bar{x}) \leq -\theta v^*(\overline{m}^*).$$

If $\bar{x} \notin S_{\mathcal{R}_0}$, then $\mathcal{F}_{\mathcal{R}_0}(\bar{x}) = \overline{m}^*$ and $\overline{m}_\theta^*$ must converge to $\overline{m}^*$, as $\theta \longrightarrow 0$. Therefore, $\lim_{\theta\to 0}[\varphi_\theta(\bar{x}) - \varphi_0(\bar{x})]/\theta = -v^*(\mathcal{F}_{\mathcal{R}_0}(\bar{x}))$, for $\bar{x} \in \Sigma \setminus S_{\mathcal{R}_0}$. Since (φ_0, ψ_0^*) is a maximizer, by the dominated convergence theorem

$$0 = \lim_{\theta\to 0} \frac{I(\varphi_\theta, \psi_\theta^*) - I(\varphi_0, \psi_0^*)}{\theta} = -\int_\Sigma v^*(\mathcal{F}_{\mathcal{R}_0}(x)) f(x)\, dx + \int_{\Sigma^*} v^*(m^*) g^*(m^*)\, dm^*.$$

By Lemma 7.1, $\mathcal{R}_0$ is a weak solution. □

According to Lemma 7.2, to study the existence of solutions of the Σ-reflector problem, we just need to consider the existence of maximizers of $I(u, v^*)$ in $\mathcal{K}$. The difficulty in variational arguments is the singularity of the cost function $c(x, m^*)$. First deal with the case $f, g \in L^p(\Sigma)$, $p > 1$.

Theorem 7.3. *Suppose that Σ is strongly convex, f, $g \geq 0$, f, $g \in L^p(\Sigma)$ with $p > 1$ such that (4-3) holds. Then there exists a maximizer (φ, ψ^*) of $I(u, v^*)$ in $\mathcal{K}$ such that $\mathcal{R} = \{e^{\psi_c^*(x)}x : x \in \Sigma\}$ is a weak solution of the Σ-reflector problem.*

Proof. Choose $(u_k, v_k^*) \in \mathcal{K}$ such that $I(u_k, v_k^*) \longrightarrow I_0 = \sup_{(u,v^*)\in\mathcal{K}} I(u, v^*)$ as $k \to \infty$. Since $I(u + C, v^* - C) = I(u, v^*)$ for any constant C by (4-3), may assume $\max_\Sigma u_k = u_k(x_k) = 0$, and furthermore, if necessary replaced by $(u_k, \max[v_k^*, \log\frac{1}{2}])$, $v_k^* \geq \log\frac{1}{2}$. We now show that there exists a constant $C_0 > 0$ such that

$$\log(1/2) \leq v_k^* \leq C_0.$$

Proceed by contradiction. Assume that $v_k^*(m_k^*) > C_0$ for some $m_k^* \in \Sigma^*$, where C_0 will be determined later. Obviously, $u_k(x) \leq c(x, m_k^*) - v_k^*(m_k^*)$ and $v_k^*(m^*) \leq c(x_k, m^*)$. Then

$$\begin{aligned} I(u_k, v_k^*) \leq & \int_\Sigma -\log(1 - m_k^* x) f(x)\, dx \\ & - C_0 \int_\Sigma f(x)\, dx - \int_{\Sigma^*} \log(1 - x_k m^*) g^*(m^*)\, dm^*. \end{aligned} \tag{7-4}$$

If we can prove $\|\log(1-m_0^*x)\|_{L^q(\Sigma)} \le C$ and $\|\log(1-x_1m^*)\|_{L^q(\Sigma^*)} \le C$ for any $x_1 \in \Sigma$, $m_0^* \in \Sigma^*$, where $q = p/(p-1)$, then by the Hölder inequality, for k and C_0 sufficiently large, (7-4) yields the following contradiction

$$I[0, \log(1/2)] - 1 \le I(u_k, v_k^*) \le C - C_0 \int_\Sigma f(x)\, dx.$$

To prove $\log(1-m_0^*x) \in L^q(\Sigma)$, set $E_j = \{x \in \Sigma : 1 - m_0^*x \le 2^{-j}\}$. Geometrically, E_j is the cross section of Σ between its tangent plane at $x_0 = \mathcal{D}(m_0^*)$ with equation $1 - m_0^*x = 0$ and the plane of equation $1 - m_0^*x = 2^{-j}$. Since Σ is strongly convex, $area(E_j) \le C2^{-j(n-1)/2}$. Therefore

$$\int_\Sigma |\log(1-m_0^*x)|^q\, dx \le C + \sum_{j=1}^{\infty} (j+1)^q area(E_j) \le C.$$

Similarly, $\log(1 - x_1m^*) \in L^q(\Sigma^*)$.

We consider the functions

$$\begin{aligned} \hat{u}_k(x) &= \inf_{m^* \in \Sigma^*} [c(x, m^*) - v_k^*(m^*)], \\ \hat{v}_k^*(m^*) &= \inf_{x \in \Sigma} [c(x, m^*) - \hat{u}_k(x)]. \end{aligned}$$

Obviously, $(\hat{u}_k, \hat{v}_k^*) \in \mathcal{K}$, $u_k \le \hat{u}_k$, and $v_k^* \le \hat{v}_k^*$. From the bounds of v_k^*, $\log\frac{1}{2} - C_0 \le \hat{u}_k \le 0$. Since v_k^* and $\hat{u}_k$ are uniformly bounded, there exists a constant $\varepsilon_0 > 0$ such that if $\hat{u}_k(x) + v_k^*(m^*) = c(x, m^*)$ then $1 - xm^* \ge 2\varepsilon_0$. Therefore, there exists $\eta_0 > 0$ such that for $x_0 \in \Sigma$ and $x \in N_{\eta_0}(x_0)$, $\hat{u}_k(x) = \inf_{m^* \in E_0^*}[c(x, m^*) - v_k^*(m^*)]$, where $E_0^* = \{m^* \in \Sigma^* : 1 - x_0m^* \ge \varepsilon_0\}$. For x_1, $x_2 \in N_{\eta_0}(x_0)$

$$|\hat{u}_k(x_2) - \hat{u}_k(x_1)| \le \sup_{m^* \in E_0^*} |c(x_2, m^*) - c(x_1, m^*)| \le C|x_2 - x_1|.$$

Similarly, we can prove that $\hat{v}_k^*$ is uniformly bounded and Lipschitz. By the Ascoli-Arzelà theorem, one may assume that $\hat{u}_k \longrightarrow \varphi$ in $C(\Sigma)$ and $\hat{v}_k^* \longrightarrow \psi^*$ in $C(\Sigma^*)$, as $k \to \infty$. Obviously, (φ, ψ^*) maximizes $I(u, v^*)$ in $\mathcal{K}$. $\square$

For the general case f, $g \in L^1(\Sigma)$, we will establish the existence of maximizers of $I(u, v^*)$ through cost function approximation, a priori estimates, and use of test function. For small $\varepsilon > 0$, consider approximate cost function on $\Sigma \times \Sigma^*$ given by

$$c_\varepsilon(x, m^*) = -\log(1 + \varepsilon - xm^*)$$

and the set of corresponding admissible functions

$$\mathcal{K}_\varepsilon = \{(u, v^*) \in C(\Sigma) \times C(\Sigma^*) : \ u(x) + v^*(m^*) \le c_\varepsilon(x, m^*) \text{ on } \Sigma \times \Sigma^*\}.$$

Lemma 7.4. *Suppose that Σ is strongly convex, $f,\, g \geq 0$, $f,\, g \in L^1(\Sigma)$ such that (4-3) holds. Then there exists a maximizer $(\varphi_\varepsilon, \psi^*_\varepsilon)$ of $I(u, v^*)$ in $\mathcal{K}_\varepsilon$ such that $\varphi_\varepsilon = (\psi^*_\varepsilon)_{c_\varepsilon}$ and $\psi^*_\varepsilon = (\varphi_\varepsilon)^{c_\varepsilon}$.*

Proof. The proof is similar to but easier than that of Theorem 7.3, since c_ε is smooth. Let $(u_k, v^*_k) \in \mathcal{K}_\varepsilon$ such that $I(u_k, v^*_k) \longrightarrow I_\varepsilon = \sup_{(u,v^*)\in\mathcal{K}_\varepsilon} I(u, v^*)$, as $k \to \infty$. Consider $\hat{u}_k = (v^*_k)_{c_\varepsilon} - C_k = (v^*_k + C_k)_{c_\varepsilon}$ and $\hat{v}^*_k = (\hat{u}_k)^{c_\varepsilon}$, where C_k is chosen such that $\sup_\Sigma \hat{u}_k(x) = 0$. Obviously, $u_k - C_k \leq \hat{u}_k$ and $v^*_k + C_k \leq \hat{v}^*_k$, and $(\hat{u}_k, \hat{v}^*_k)$ is also a maximizing sequence of $I(u, v^*)$ in $\mathcal{K}_\varepsilon$. For $x_1,\, x_2 \in \Sigma$

$$|\hat{u}_k(x_2) - \hat{u}_k(x_1)| \leq \sup_{m^*\in\Sigma^*} |c_\varepsilon(x_2, m^*) - c_\varepsilon(x_1, m^*)| \leq C(\varepsilon)|x_2 - x_1|.$$

Therefore, $\hat{u}_k$ and $\hat{v}^*_k$ are uniformly bounded and Lipschitz. It follows that $I(u, v^*)$ can be maximized in $\mathcal{K}_\varepsilon$. As in the proof of Lemma 7.2 (ii), if $(\widehat{\varphi}_\varepsilon, \widehat{\psi}^*_\varepsilon)$ maximizes $I(u, v^*)$ in $\mathcal{K}_\varepsilon$, then $(\varphi_\varepsilon, \psi^*_\varepsilon) = ((\widehat{\psi}^*_\varepsilon)_{c_\varepsilon},\ [(\widehat{\psi}^*_\varepsilon)_{c_\varepsilon}]^{c_\varepsilon})$ is also a maximizer such that $\varphi_\varepsilon = (\psi^*_\varepsilon)_{c_\varepsilon}$ and $\psi^*_\varepsilon = (\varphi_\varepsilon)^{c_\varepsilon}$. $\square$

Lemma 7.5. *Let $(\varphi_\varepsilon, \psi^*_\varepsilon)$ be a maximizer of $I(u, v^*)$ in $\mathcal{K}_\varepsilon$ such that $\varphi_\varepsilon = (\psi^*_\varepsilon)_{c_\varepsilon}$, $\psi^*_\varepsilon = (\varphi_\varepsilon)^{c_\varepsilon}$, and $\min_\Sigma \varphi_\varepsilon = 0$. Then $\sup_\Sigma |\varphi_\varepsilon| + \sup_{\Sigma^*} |\psi^*_\varepsilon| \leq C$, where the constant C is independent of ε.*

Proof. To simplify notations, we will use (φ, ψ^*) to replace $(\varphi_\varepsilon, \psi^*_\varepsilon)$ in the proof. For $y \in \Sigma$ and $E \subset \Sigma$, let $\mathcal{T}_\varphi(y) = \{m^* \in \Sigma^* : \varphi(y) = c_\varepsilon(y, m^*) - \psi^*(m^*)\}$ and $\mathcal{T}_\varphi(E) = \cup_{y\in E}\mathcal{T}_\varphi(y)$. Assume $\varphi(x_0) = \min_\Sigma \varphi = 0$. If $m^*_0 \in \mathcal{T}_\varphi(x_0)$, then

$$c_\varepsilon(x_0, m^*_0) - \psi^*(m^*_0) = 0 \leq \varphi(-\mathcal{D}(m^*_0)) \leq c_\varepsilon(-\mathcal{D}(m^*_0), m^*_0) - \psi^*(m^*_0).$$

It yields $x_0 = -\mathcal{D}(m^*_0)$ and $\psi^*(m^*_0) = \log\frac{1}{2+\varepsilon}$. Let $\eta > 0$ be a small number to be determined later, and consider $E = \Sigma \setminus N_\eta(\mathcal{D}(m^*_0))$. We now show the following

Claim: $\mathcal{T}_\varphi(E) \cap [\Sigma^* \setminus N_\eta(m^*_0)] \neq \emptyset$.

Prove by contradiction. Assume $\mathcal{T}_\varphi(E) \subset N_\eta(m^*_0)$. Choose $h^* \in C(\Sigma^*)$ such that $0 \leq h^* \leq 1$ on Σ^*, $h^* = 1$ on $N_{2\eta}(m^*_0)$, and $h^* = 0$ outside $N_{4\eta}(m^*_0)$. For small $\tau > 0$, define $\psi^*_\tau(m^*) = \psi^*(m^*) - \tau h^*(m^*)$ and $\varphi_\tau(x) = (\psi^*_\tau)_{c_\varepsilon}(x) = \inf_{\Sigma^*}[c_\varepsilon(x, m^*) - \psi^*_\tau(m^*)]$. Obviously, $0 \leq \varphi_\tau - \varphi \leq \tau$ on Σ.

We show that there exists $\tau_0 = \tau_0(\varepsilon) > 0$ such that for any $y_0 \in E$ and $\tau \leq \tau_0$

$$\varphi_\tau(y_0) = c_\varepsilon(y_0, m^*_\tau) - \psi^*_\tau(m^*_\tau) \implies m^*_\tau \in N_{2\eta}(m^*_0). \tag{7-5}$$

Otherwise, there exist $x_j \in E$, $\tau_j \to 0$, and $m^*_{\tau_j} \notin N_{2\eta}(m^*_0)$ such that

$$\varphi_{\tau_j}(x_j) = c_\varepsilon(x_j, m^*_{\tau_j}) - \psi^*_{\tau_j}(m^*_{\tau_j}). \tag{7-6}$$

One may assume that $x_j \to x_\infty \in E$ and $m^*_{\tau_j} \to m^*_\infty$. By taking limit in (7-6), we obtain $m^*_\infty \in \mathcal{T}_\varphi(x_\infty) \subset \mathcal{T}_\varphi(E) \subset N_\eta(m^*_0)$, which contradicts with $m^*_\infty \notin N_{2\eta}(m^*_0)$.

By (7-5) and the properties of h^*, given $x \in E$ and $\tau \leq \tau_0$, there exists $m^*_\tau \in N_{2\eta}(m^*_0)$ such that $\varphi_\tau(x) = c_\varepsilon(x, m^*_\tau) - \psi^*_\tau(m^*_\tau) \geq \varphi(x) + \tau$. Thus, $\varphi_\tau = \varphi + \tau$ on E for $\tau \leq \tau_0$. Since (φ, ψ^*) is a maximizer

$$\left(\int_{\Sigma\setminus E} + \int_E\right)\frac{\varphi_\tau - \varphi}{\tau} f\,dx + \int_{\Sigma^*}\frac{\psi^*_\tau - \psi^*}{\tau}\, g^*\,dm^* \leq 0, \qquad 0 < \tau \leq \tau_0.$$

It gives rise to the following

$$\int_E f\,dx \leq \int_{N_{4\eta}(m^*_0)} g^*\,dm^*,$$

which is a contradiction if η is small enough. Therefore, the Claim is proved.

By the Claim, there exists $x_1 \in E$ and $m^*_1 \in \mathcal{T}_\varphi(x_1) \setminus N_\eta(m^*_0)$. Clearly

$$0 \leq \varphi(x_1) \leq c_\varepsilon(x_1, m^*_0) + \log(2+\varepsilon) \leq C.$$

We have $\psi^*(m^*_1) = c_\varepsilon(x_1, m^*_1) - \varphi(x_1) \geq -C$. Since $\varphi \geq 0$, $\psi^*(m^*_1) \leq \inf_\Sigma c_\varepsilon(x, m^*_1) \leq \log\frac{1}{2}$. Therefore, we obtain

$$0 \leq \varphi(x) \leq \min\{c_\varepsilon(x, m^*_0) + \log(2+\varepsilon),\ c_\varepsilon(x, m^*_1) - \psi^*(m^*_1)\} \leq C.$$

It follows from $\psi^* = \varphi^{c_\varepsilon}$ that ψ^* is also uniformly bounded. $\square$

Theorem 7.6. *Suppose that Σ is strongly convex, f, $g \geq 0$, f, $g \in L^1(\Sigma)$ such that (4-3) holds. Then there exists a maximizer (φ, ψ^*) of $I(u, v^*)$ in $\mathcal{K}$ such that $\varphi = \psi^*_c$, $\psi^* = \varphi^c$, and $\mathcal{R} = \{e^{\varphi(x)}x : x \in \Sigma\}$ is a weak solution of the Σ-reflector problem.*

Proof. By Lemma 7.4, for small $\varepsilon > 0$, there exists $(\varphi_\varepsilon, \psi^*_\varepsilon)$ maximizing $I(u, v^*)$ in $\mathcal{K}_\varepsilon$ and satisfying $\varphi_\varepsilon = (\psi^*_\varepsilon)_{c_\varepsilon}$ and $\psi^*_\varepsilon = (\varphi_\varepsilon)^{c_\varepsilon}$. Without loss of generality, assume $\min_\Sigma \varphi_\varepsilon = 0$. φ_ε and ψ^*_ε are uniformly bounded by Lemma 7.5. As in the proof of Theorem 7.3, we can show that φ_ε and ψ^*_ε are uniformly Lipschitz. Then there exist $\varepsilon_j \longrightarrow 0$, $\varphi_{\varepsilon_j} \longrightarrow \varphi$ uniformly on Σ, and $\psi^*_{\varepsilon_j} \longrightarrow \psi^*$ uniformly on Σ^*, as $j \longrightarrow \infty$. It is easy to verify that $\varphi = \psi^*_c$ and $\psi^* = \varphi^c$. To finish the proof, we show that(φ, ψ^*) maximizes $I(u, v^*)$ in $\mathcal{K}$. Given $(u, v^*) \in \mathcal{K}$, let $u(x) + v^*(m^*) \leq M$. For $0 < \varepsilon < e^{-M}/2$, if $1 - xm^* < e^{-M}/2$, then $u(x) + v^*(m^*) \leq M \leq c_\varepsilon(x, m^*)$; if $1 - xm^* \geq e^{-M}/2$, then $u(x) + v^*(m^*) \leq c(x, m^*) \leq c_\varepsilon(x, m^*) + 2e^M\varepsilon$. Therefore, for sufficiently large j, $I(u - 2e^M\varepsilon_j, v^*) \leq I(\varphi_{\varepsilon_j}, \psi^*_{\varepsilon_j})$. We then obtain $I(u, v^*) \leq I(\varphi, \psi^*)$. $\square$

Finally, we briefly mention the connection with the optimal mass transport problem. Consider to find an optimal plan to minimize the cost functional

$$\int_\Sigma c(x, s(x)) f(x)\,dx$$

in the set $\mathcal{S}$ of all plans s, where a plan s is a measurable mapping from Σ to Σ^* such that it is measure-preserving:

$$\int_{s^{-1}(E^*)} f\,dx = \int_{E^*} g^*\,dm^*, \quad \forall \text{ Borel set } E^* \subset \Sigma^*,$$

or equivalently

$$\int_{\Sigma} v^*(s(x))f(x)\,dx = \int_{\Sigma^*} v^*(m^*)g^*(m^*)\,dm^*, \quad \forall\ v^* \in C(\Sigma^*).$$

It can be shown that if (φ, ψ^*) is a maximizer of $I(u, v^*)$ in $\mathcal{K}$ such that $\varphi = \psi^*_c$ and $\psi^* = \varphi^c$, then $\mathcal{F}_{\mathcal{R}}$ is an optimal plan minimizing the cost, where $\mathcal{R}$ is parameterized by $e^{\varphi(x)}x$. If $\int_G f\,dx > 0$ for any open set $G \subset \Sigma$, the uniqueness of solutions can be treated through this connection. These discussions are similar to those in [Wan04]. We leave the details to readers.

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