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On the regularity of reflector antennas

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1. Introduction

By the Snell law of reflection, a light ray incident upon a reflective surface will be reflected at an angle equal to the incident angle. Both angles are measured with respect to the normal to the surface. If a light ray emanates from O in the direction $x \in S^{n-1}$, and $\mathcal{A}$ is a perfectly reflecting surface, then the reflected ray has direction:

$$x^* = T(x) = x - 2\,\langle x, \nu \rangle\, \nu, \tag{1.1}$$

where ν is the outer normal to $\mathcal{A}$ at the point where the light ray hits $\mathcal{A}$.

Suppose that we have a light source located at O, and Ω, Ω^* are two domains in the sphere S^{n-1}, $f(x)$ is a positive function for $x \in \Omega$ (input illumination intensity), and $g(x^*)$ is a positive function for $x^* \in \Omega^*$ (output illumination intensity). If light emanates from O with intensity $f(x)$ for $x \in \Omega$, the *far field reflector antenna problem* is to find a perfectly reflecting surface $\mathcal{A}$ parametrized by $z = \rho(x)\,x$ for $x \in \Omega$, such that all reflected rays by $\mathcal{A}$ fall in the directions in Ω^*, and the output illumination received in the direction x^* is $g(x^*)$; that is, $T(\Omega) = \Omega^*$, where T is given by (1.1). Assuming there is no loss of energy in the reflection, then by the law of conservation of energy

$$\int_\Omega f(x)\,dx = \int_{\Omega^*} g(x^*)\,dx^*.$$

In addition, and again by conservation of energy, the map T defined by (1.1) is measure-preserving:

$$\int_{T^{-1}(E)} f(x)\,dx = \int_E g(x^*)\,dx^*, \qquad \text{for all } E \subset \Omega^* \text{ Borel set},$$

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and consequently, the Jacobian of T is $\dfrac{f(x)}{g(T(x))}$. It yields the following nonlinear equation on S^{n-1} (see [GW98]):

$$\frac{\det\left(\nabla_{ij}u+(u-\eta)e_{ij}\right)}{\eta^{n-1}\det(e_{ij})}=\frac{f(x)}{g(T(x))}, \tag{1.2}$$

where $u=1/\rho$, $\nabla =$ covariant derivative, $\eta=\dfrac{|\nabla u|^2+u^2}{2u}$, and e is the metric on S^{n-1}. This very complicated fully nonlinear PDE of Monge-Ampère type received attention from the engineering and numerical points of view because of its applications [Wes83]. From the point of view of the theory of nonlinear PDEs, the study of this equation began only recently with the notion of weak solution introduced by Xu-Jia Wang [Wan96] and by L. Caffarelli and V. Oliker [CO94], [Oli02].

The reflector antenna problem in the case $n=3$, $\Omega\subset S^2_+$, and $\Omega^*\subset S^2_-$, where S^2_+ and S^2_- are the northern and southern hemispheres respectively, was discussed in [Wan96], [Wan04]. The existence and uniqueness up to dilations of weak solutions were proved in [Wan96] if f and g are bounded away from 0 and ∞. Regularity of weak solutions was also addressed in [Wan96] and it was proved that weak solutions are smooth if f, g are smooth and Ω, Ω^* satisfy certain geometric conditions. Xu-Jia Wang [Wan04] recently discovered that this antenna problem is an optimal mass transportation problem on the sphere for the cost function $c(x,y)=-\log(1-x\cdot y)$; see also [GO03].

On the other hand, the global reflector antenna problem (i.e., $\Omega=\Omega^*=S^{n-1}$) was treated in [CO94], [GW98]. When f and g are strictly positive bounded, the existence of weak solutions was established in [CO94] and the uniqueness up to homothetic transformations was proved in [GW98]. If f, $g\in C^{1,1}(S^{n-1})$, Pengfei Guan and Xu-Jia Wang [GW98] showed that weak solutions are $C^{3,\alpha}$ for any $0<\alpha<1$. Actually, slightly more general results were discussed in these references.

We mention that in the case of two reflectors a connection with mass transportation was found by T. Glimm and V. Oliker [GO04].

It is noted that the reflector antenna problem is somehow analogous to the Monge-Ampère equation, however, it is more nonlinear in nature and more difficult than the Monge-Ampère equation.

Our purpose in this paper is to establish some important quantitative and qualitative properties of weak solutions to the global antenna problem, that is, when $\Omega=\Omega^*=S^{n-1}$. Three important results are crucial for the regularity theory of weak solutions to the Monge-Ampère equation: interior gradient estimates, the Alexandrov estimate, and Caffarelli's strict convexity. Our first goal here is to extend these fundamental estimates to the setting of the reflector antenna problem. This is contained in Theorems 3.3–3.5. In our case these estimates are much more complicated to establish than the counterpart for convex functions due to the lack of the affine invariance property of the equation (1.2) and the fact that the geometry of cofocused paraboloids is much more complicated than that of planes. Our second goal is to prove the counterpart of Caffarelli's strict convexity result in this setting, Theorem 4.2. Finally, the third goal is to show that weak solutions to the global reflector antenna problem are C^1 under the assumption that input and output illumination intensities are strictly positive bounded. To this end, in Section 5 we establish some properties of the Legendre transforms of weak solutions and combine them together with Theorem 4.2 to obtain the desired regularity.

2. Preliminaries

Let $\mathcal{A}$ be an antenna parametrized by $y = \rho(x)\, x$ for $x \in S^{n-1}$. Throughout this paper, we assume that there exist r_1, r_2 such that

$$0 < r_1 \leq \rho(x) \leq r_2, \qquad \forall x \in S^{n-1}. \tag{2.1}$$

Given $m \in S^{n-1}$ and $b > 0$, $P(m, b)$ denotes the paraboloid of revolution in $\mathbb{R}^n$ with focus at 0, axis m, and directrix hyperplane $\Pi(m, b)$ of equation $m \cdot y + 2b = 0$. The equation of $P(m, b)$ is given by $|y| = m \cdot y + 2b$. If $P(m', b')$ is another such paraboloid, then $P(m, b) \cap P(m', b')$ is contained in the bisector of the directrices of both paraboloids, denoted by $\Pi[(m, b), (m', b')]$, and that has equation $(m - m') \cdot y + 2(b - b') = 0$; see Figure 1.

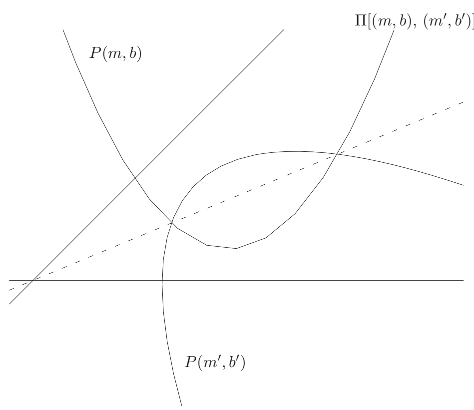

Figure 1

LEMMA 2.1. *Let $P(e_n, a)$ and $P(m, b)$ be two paraboloids with $m = (m', m_n)$. Then the projection onto $\mathbb{R}^{n-1}$ of $P(e_n, a) \cap P(m, b)$ is a sphere $S_{a,b,m}$ with equation*

$$S_{a,b,m} \equiv \left| x' - 2\,a \frac{m'}{1 - m_n} \right|^2 = \frac{8ab}{1 - m_n} = R^2_{a,b,m}.$$

Proof. Since $P(e_n, a)$ has focus at 0, it follows that it has equation $x_n = \frac{1}{4a}|x'|^2 - a$. The intersection of $P(e_n, a)$ and $P(m, b)$ is contained in the hyperplane of equation $(m - e_n) \cdot x + 2(b - a) = 0$. Hence the equation of $\Pi[(e_n, a), (m, b)]$ can be written as

$$x_n = \frac{m' \cdot x'}{1 - m_n} + 2\frac{b - a}{1 - m_n}.$$

Therefore the points $x = (x', x_n) \in P(e_n, a) \cap P(m, b)$ satisfy the equation

$$\frac{1}{4a}|x'|^2 - a = \frac{m' \cdot x'}{1 - m_n} + 2\frac{b - a}{1 - m_n},$$

which simplifies to the sphere in $\mathbb{R}^{n-1}$

$$S_{a,b,m} \equiv \left| x' - 2\,a\frac{m'}{1 - m_n} \right|^2 = \frac{8a(b - a)}{1 - m_n} + 4a^2 \left(1 + \left(\frac{|m'|}{1 - m_n} \right)^2 \right) = R^2_{a,b,m}.$$

Since $|m'|^2 + m_n^2 = 1$, a direct simplification yields

$$R^2_{a,b,m} = \frac{8ab}{1 - m_n}. \qquad \square$$

Definition 2.2 (Supporting paraboloid). We say that $P(m, b)$ is a *supporting paraboloid* to the antenna $\mathcal{A}$ at the point $y \in \mathcal{A}$, or that $P(m, b)$ *supports* $\mathcal{A}$ at the point $y \in \mathcal{A}$, if $y \in P(m, b)$ and $\mathcal{A}$ is contained in the interior region limited by the surface described by $P(m, b)$.

Definition 2.3 (Admissible antenna). The antenna $\mathcal{A}$ is *admissible* if it has a supporting paraboloid at each point.

Remark 2.4. We remark that if $P(m, b)$ is a supporting paraboloid to the antenna $\mathcal{A}$, then $r_1 \leq b \leq r_2$. To prove it, assume that $P(m, b)$ contacts $\mathcal{A}$ at $\rho(x_0)x_0$ for $x_0 \in S^{n-1}$. Obviously, $0 < b \leq \rho(x_0) \leq r_2$ by (2.1). On the other hand, $b \geq \rho(-m) \geq r_1$ also by (2.1).

Definition 2.5 (Reflector map). Given an admissible antenna $\mathcal{A}$ parametrized by $z = \rho(x)\,x$ and $y \in S^{n-1}$, the *reflector mapping* associated with $\mathcal{A}$ is

$$\mathcal{N}_{\mathcal{A}}(y) = \{m \in S^{n-1} : P(m, b) \text{ supports } \mathcal{A} \text{ at } \rho(y)\,y\}.$$

If $E \subset S^{n-1}$, then $\mathcal{N}_{\mathcal{A}}(E) = \cup_{y \in E} \mathcal{N}_{\mathcal{A}}(y)$.

Obviously, $\mathcal{N}_{\mathcal{A}}$ is the generalization of the mapping T in (1.1) for nonsmooth antennas. The set $\cup_{y_1 \neq y_2}[\mathcal{N}_{\mathcal{A}}(y_1) \cap \mathcal{N}_{\mathcal{A}}(y_2)]$ has measure 0, and as a consequence, the class of sets $E \subset S^{n-1}$ for which $\mathcal{N}_{\mathcal{A}}(E)$ is Lebesgue measurable is a Borel σ-algebra; see [Wan96, Lemma 1.1]. The notion of weak solution can be introduced through energy conservation in two ways. The first one is the natural one and uses $\int_{\mathcal{N}_{\mathcal{A}}^{-1}(E^*)} f\,dx = \int_{E^*} g\,dm$, through $\mathcal{N}_{\mathcal{A}}^{-1}$. And the second one uses $\int_E f\,dx = \int_{\mathcal{N}_{\mathcal{A}}(E)} g\,dm$, through $\mathcal{N}_{\mathcal{A}}$. For nonnegative functions f, $g \in L^1(S^{n-1})$, it is easy to show using [Wan96, Lemma 1.1] that these two ways are equivalent. We will use the second way to define weak solutions. Given $g \in L^1(S^{n-1})$ we define the Borel measure

$$\mu_{g,\mathcal{A}}(E) = \int_{\mathcal{N}_{\mathcal{A}}(E)} g(m)\,dm.$$

Definition 2.6 (Weak solution). The surface $\mathcal{A}$ is a *weak solution of the antenna problem* if $\mathcal{A}$ is admissible and

$$\mu_{g,\mathcal{A}}(E) = \int_E f(x)\,dx,$$

for each Borel set $E \subset S^{n-1}$.

By the definition, smooth solutions to (1.2) are weak solutions. If $C\mathcal{A}$ is the C-dilation of $\mathcal{A}$ with respect to O, then $N_{C\mathcal{A}} = N_{\mathcal{A}}$. Therefore, any dilation of a weak solution is also a weak solution of the same antenna problem.

We make a remark on (2.1). If the input intensity f and the output intensity g are bounded away from 0 and ∞, and $\mathcal{A}$ is normalized with $\inf_{s\in S^{n-1}} \rho(x) = 1$, then there exists $r_0 > 0$ such that $\sup_{x\in S^{n-1}} \rho(x) \le r_0$, by [GW98].

3. Estimates for reflector mapping

Throughout this paper, we assume that f and g are bounded away from 0 and ∞, and there exist positive constants in λ, Λ such that

$$\lambda\,|E| \le |\mathcal{N}_{\mathcal{A}}(E)| \le \Lambda\,|E|, \tag{3.1}$$

for all Borel subsets $E \subset S^{n-1}$.

Let $\mathcal{A}$ be an admissible antenna and $P(m, b_0)$ a paraboloid focused at O such that $\mathcal{A} \cap P(m, b_0) \neq \emptyset$. Let $\mathcal{S}_{\mathcal{A}}(P(m, b_0))$ be the portion of $\mathcal{A}$ cut by $P(m, b_0)$ and lying outside $P(m, b_0)$, that is,

$$\mathcal{S}_{\mathcal{A}}(P(m, b_0)) = \{z \in \mathcal{A} :\, \exists\, b \ge b_0 \text{ such that } z \in P(m, b)\}. \tag{3.2}$$

$\mathcal{S}_{\mathcal{A}}(P(m, b_0))$ can be viewed as a level set or cross section of the reflector antenna $\mathcal{A}$.

We shall first establish some estimates for the reflector mapping on cross sections of the antenna $\mathcal{A}$.

3.1. *Projections of cross sections.* We begin with a geometric lemma concerning the convexity of projections of cross sections of $\mathcal{A}$.

Lemma 3.1. *Let $\mathcal{A}$ be an admissible antenna and let $P(e_n, a)$ be a paraboloid focused at 0 such that $P(e_n, a) \cap \mathcal{A} \neq \emptyset$. Then*

(a) *If $x_0, x_1 \in \mathcal{S}_{\mathcal{A}}(P(e_n, a))$, then there exists a planar curve $\mathcal{C} \subset \mathcal{S}_{\mathcal{A}}(P(e_n, a))$ joining x_0 and x_1.*

(b) *Let $\mathcal{R} = \mathcal{S}_{\mathcal{A}}(P(e_n, a))$ and $\mathcal{R}'$ be the projection of $\mathcal{R}$ onto $\mathbb{R}^{n-1}$ which is identified as a hyperplane in $\mathbb{R}^n$ through O with the normal e_n. Then $\mathcal{R}'$ is convex.*

Proof. Let x_0', x_1' be the projection of x_0, x_1 onto $\mathbb{R}^{n-1}$, and let L be the 2-dimensional plane through x_0', x_1' and parallel to e_n. Consider the planar curve $L \cap \mathcal{A}$ that contains x_0, x_1. We claim that the lower portion of $L \cap \mathcal{A}$ connecting x_0, x_1 lies below $P(e_n, a)$. Indeed, let x be on this lower portion of $L \cap \mathcal{A}$ and let $P(m, b)$ be a supporting paraboloid to $\mathcal{A}$ at the point x. If $m = e_n$, then $a \leq b$ and x is below $P(e_n, a)$. Now consider the case $m \neq e_n$. Obviously, the points x_0, x_1 are below $P(e_n, a)$ and inside $P(m, b)$. Therefore, x_0, x_1 lie below the bisector $\Pi[(e_n, a), (m, b)]$ and hence below the line $L \cap \Pi[(e_n, a), (m, b)]$. Since $L \cap \mathcal{A}$ is a convex curve, it follows that the lower portion of $L \cap \mathcal{A}$ connecting x_0 and x_1 lies below $L \cap \Pi[(e_n, a), (m, b)]$ and so does x. It implies that x is below $P(e_n, a)$. This proves (a) and as a result part (b) follows. □

Remark 3.2. Throughout this section we use the following construction. If $P(e_n, a) \cap \mathcal{A} \neq \emptyset$, $\mathcal{R} = \mathcal{S}_{\mathcal{A}}(P(e_n, a))$, and $\mathcal{R}'$ is the projection of $\mathcal{R}$ onto $\mathbb{R}^{n-1}$ parallel to the directrix hyperplane $\Pi(e_n, a)$, then E will denote the Fritz John $(n-1)$-dimensional ellipsoid of $\mathcal{R}'$; that is, $\frac{1}{n-1} E \subset \mathcal{R}' \subset E$; we assume that E has principal axes $\lambda_1, \cdots, \lambda_{n-1}$ in the coordinate directions $e_1, \cdots, e_{n-1}$.

3.2. *Estimates in case the diameter of E is big.* For a convex function $v(x)$ on a convex domain Ω, it is well known that $|Dv(x)| \leq C \operatorname{osc}_{\Omega} v / \operatorname{dist}(x, \partial\Omega)$, for any $x \in \Omega$, see [Gut01, Lemma 3.2.1]. This fact gives rise to an estimate from above of the measure of the image of the norm mapping. The following theorem extends this result to the setting of the reflector mapping.

Theorem 3.3. *Let $\mathcal{A}$ be an admissible antenna satisfying* (2.1) *and let $P(e_n, a + h)$ with $h > 0$ small be a supporting paraboloid to $\mathcal{A}$. Denote by $\mathcal{R} = \mathcal{S}_{\mathcal{A}}(P(e_n, a))$ the portion of $\mathcal{A}$ bounded between $P(e_n, a+h)$ and $P(e_n, a)$, and let $\mathcal{R}'$ and E be defined as in Remark* 3.2. *Let $\mathcal{R}_{1/2}$ be the lower portion of $\mathcal{R}$ whose projection onto $\mathbb{R}^{n-1}$ is $\frac{1}{2(n-1)} E$.*

(a) *Assume $d_1 \leq d = \operatorname{diam}(E) \leq d_2$. If $P(m, b)$ is a supporting paraboloid to $\mathcal{A}$ at some $Q \in \mathcal{R}_{1/2}$ with $m = (m', m_n) = (m_1, \cdots, m_{n-1}, m_n)$, then $|m_i| \leq Ch/\lambda_i$ for $i = 1, \cdots, n-1$, and $|m'| \leq \sqrt{2}\sqrt{1 - m_n} \leq C\sqrt{h}/d$, where C depends only on structural constants, d_1, and d_2.*

(b) *Assume that* $\dfrac{\sqrt{h}}{d} \leq \eta_0$ *with* η_0 *small. Let* $\rho^{-1}(\mathcal{R}_{1/2})$ *be the preimage of* $\mathcal{R}_{1/2}$ *on* S^{n-1}*. Then* $\mathcal{N}_{\mathcal{A}}(\rho^{-1}(\mathcal{R}_{1/2})) \subset \{(m', m_n) \in S^{n-1} : \sqrt{1-m_n} \leq C\sqrt{h}/d\}$ *and*

$$|\mathcal{N}_{\mathcal{A}}(\rho^{-1}(\mathcal{R}_{1/2}))| \leq C \prod_{i=1}^{n-1} \min\left\{\frac{\sqrt{h}}{d}, \frac{h}{\lambda_i}\right\},$$

where C *depends only on structural constants and* η_0*.*

Proof. Suppose that $P(m, b)$ is a supporting paraboloid to $\mathcal{A}$ at some point $Q \in \mathcal{R}_{1/2}$. Let $\tau = \dfrac{m'}{|m'|} \in \mathbb{R}^{n-1}$, $m_\tau = |m'|$, and write

$$m = (m_\tau\, \tau, m_n). \tag{3.3}$$

We have $1 = |m|^2 = m_n^2 + m_\tau^2$ and therefore

$$m_\tau^2 \leq 2\,(1 - m_n). \tag{3.4}$$

From Lemma 2.1, the points $x = (x', x_n) \in P(e_n, a) \cap P(m, b)$ satisfy the equation

$$S_{a,b,m} \equiv \left| x' - 2\,a \frac{m_\tau}{1 - m_n}\, \tau \right|^2 = R^2_{a,b,m},$$

with

$$R^2_{a,b,m} = \frac{8ab}{1 - m_n}.$$

Our goal now is to estimate the reflector mapping over the interior lower portion $\mathcal{R}_{1/2}$ whose projection on $\mathbb{R}^{n-1}$ is $\frac{1}{2(n-1)}E$.

Recall Remark 2.4 and that h is very small. Let Q' denote the projection of Q in the direction e_n; that is, $Q' \in \frac{1}{2(n-1)}E$. We may assume $m \neq e_n$. Obviously, there exists $0 < \varepsilon_0 \leq 1$ such that $Q \in P(e_n, a + \varepsilon_0 h) \cap P(m, b)$; see Figure 2. Let $\mathcal{P}$ be the portion of $P(m, b)$ below $\mathcal{R}$ and defined over $\mathcal{R}'$. Since $P(e_n, a + \varepsilon_0 h) \cap P(m, b) \subset \Pi[(e_n, a + \varepsilon_0 h), (m, b)]$, it follows that $\mathcal{P}$ crosses $\Pi[(e_n, a + \varepsilon_0 h), (m, b)]$ and $P(e_n, a + \varepsilon_0 h)$. Let $S_{a+\varepsilon_0 h, b, m}$ be the sphere from Lemma 2.1 obtained projecting $\Pi[(e_n, a + \varepsilon_0 h), (m, b)] \cap P(m, b)$ on $\mathbb{R}^{n-1}$, and let $B_{a+\varepsilon_0 h, b, m}$ be the solid ball whose boundary is $S_{a+\varepsilon_0 h, b, m}$. Since $\Pi[(e_n, a + \varepsilon_0 h), (m, b)]$ traverses $P(m, b)$, it follows that $\mathcal{P}$ is below the bisector $\Pi[(e_n, a + \varepsilon_0 h), (m, b)]$ in the region $\mathcal{R}' \cap B_{a+\varepsilon_0 h, b, m}$, and therefore $\mathcal{P}$ is below $P(e_n, a + \varepsilon_0 h)$ in the same region. Therefore, $\mathcal{P}$ is above (or inside) $P(e_n, a + \varepsilon_0 h)$ in $\mathcal{R}' \setminus B_{a+\varepsilon_0 h, b, m}$.

For $x = (x', x_n) \in \mathcal{P}$ with $x' \in \mathcal{R}' \setminus B_{a+\varepsilon_0 h, b, m}$, x must be between $P(e_n, a)$ and $P(e_n, a + \varepsilon_0 h)$. Hence there exists $\varepsilon = \varepsilon_x$ such that $0 \leq \varepsilon \leq \varepsilon_0$ with

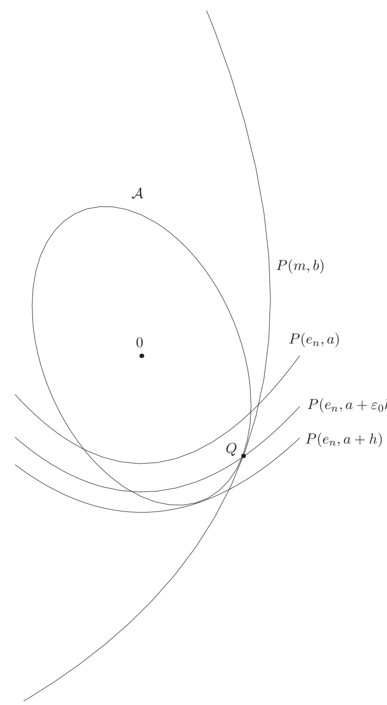

Figure 2

$x \in P(e_n, a+\varepsilon h) \cap P(m,b)$. Consequently, $x' \in S_{a+\varepsilon h,b,m}$ and from Lemma 2.1 we have

$$\left| x' - 2(a+\varepsilon h)\frac{m_\tau}{1-m_n}\tau \right|^2 = \frac{8(a+\varepsilon h)b}{1-m_n} = R^2_{a+\varepsilon h,b,m}.$$

On the other hand, x' is outside $S_{a+\varepsilon_0 h,b,m}$. It follows that

$$\begin{aligned}
\sqrt{\frac{8(a+\varepsilon_0 h)b}{1-m_n}} &\leq \left| x' - 2(a+\varepsilon_0 h)\frac{m_\tau}{1-m_n}\tau \right| \\
&\leq \left| x' - 2(a+\varepsilon h)\frac{m_\tau}{1-m_n}\tau \right| + 2(\varepsilon_0 - \varepsilon)h\frac{m_\tau}{1-m_n} \\
&\leq \sqrt{\frac{8(a+\varepsilon h)b}{1-m_n}} + 2\sqrt{2}\frac{h}{\sqrt{1-m_n}} \\
&\leq (1+Ch)\sqrt{\frac{8(a+\varepsilon_0 h)b}{1-m_n}}.
\end{aligned}$$

One then obtains that $\mathcal{R}' \setminus B_{a+\varepsilon_0 h,b,m}$ is contained in a ring with inner radius $R = R_{a+\varepsilon_0 h,b,m}$ and width $C\,R\,h$. Since the inner sphere of the ring $S_{a+\varepsilon_0 h,b,m}$ passes through $Q' \in \frac{1}{2(n-1)}E$, its tangent at Q' traverses $\frac{1}{2(n-1)}E$ and the ring.

Thus, there exists an ellipsoid $E_0 \subset \mathcal{R}' \setminus B_{a+\varepsilon_0 h,b,m}$ whose axes are comparable and parallel to those of E. Moreover, E_0 is contained in a cylinder $\mathbb{C}$ whose height is $C\,R\,h$ and whose base is an $(n-2)$-dimensional ball with radius $CR\sqrt{h}$ and center Q'. Since $\mathrm{diam}(\mathbb{C}) = CR\sqrt{h}$, one obtains that

$$d \le CR\sqrt{h} \quad \text{and therefore} \quad \sqrt{1-m_n} \le C\sqrt{h}/d. \tag{3.5}$$

As $\sqrt{h}/d$ is small, m_n is close to 1 and R is very large. From (3.4) and (3.5) we obtain the estimate $|m_\tau| \le C\sqrt{h}/d$.

Let x_0' be the center of E_0 and E_C be the center of E. We want to show that

$$\left| \left(m' - \frac{1-m_n}{2a} E_C \right) \cdot \overrightarrow{x_0' x'} \right| \le C\,h, \tag{3.6}$$

for all $x' \in E_0$. For simplicity, let $C_0 = 2(a+\varepsilon_0 h)\dfrac{m_\tau}{1-m_n}\tau$ be the center of the ring. We claim that the angle between $\overrightarrow{C_0E_C}$ and the radial direction $\overrightarrow{C_0Q'}$ is very small; that is, $\mathrm{angle}(\overrightarrow{C_0E_C}, \overrightarrow{C_0Q'}) \le C\,d/R$. In fact, by the law of cosines, we have that

$$|\overrightarrow{E_CQ'}|^2 = |\overrightarrow{C_0Q'}|^2 + |\overrightarrow{C_0E_C}|^2 - 2\overrightarrow{C_0Q'} \cdot \overrightarrow{C_0E_C}.$$

Without loss of generality, we may assume that $|\overrightarrow{C_0E_C}| \le |\overrightarrow{C_0Q'}|$. If we set $\overrightarrow{C_0Q'} = |\overrightarrow{C_0Q'}|\tau_r = R_1\tau_r$, $A_1 = |\overrightarrow{E_CQ'}|$, and $\overrightarrow{C_0E_C} = |\overrightarrow{C_0E_C}|\tau_E = (R_1 - A_2)\tau_E$, where $0 < A_2 \le A_1 \le d$, then

$$A_1^2 = R_1^2 + (R_1 - A_2)^2 - 2R_1(R_1 - A_2)\,\tau_r \cdot \tau_E.$$

Since R is large and $\frac{R_1}{R} \approx C$ by (3.5), we get the following

$$1 - \tau_r \cdot \tau_E = \frac{A_1^2 - A_2^2}{2R_1(R_1 - A_2)} \le \frac{Cd^2}{R^2},$$

and the claim is proved.

Continuing with the proof of (3.6), write $\tau_E = k_r\tau_r + k_t\tau_t$, where τ_t is a unit vector in the tangent plane of the sphere $S_{a+\varepsilon_0 h,b,m}$ at the point Q'; that is, $\tau_t \perp \tau_r$, and $k_t \ge 0$. Therefore, we have

$$\begin{aligned} \tau_E \cdot \tau_t = k_t &= \sqrt{1 - (\tau_r \cdot \tau_E)^2} \\ &= \sqrt{(1 + \tau_r \cdot \tau_E)(1 - \tau_r \cdot \tau_E)} \\ &\le \sqrt{2\frac{Cd^2}{R^2}} \le C\frac{d}{R}. \end{aligned}$$

For $x', x'' \in E_0$, write

$$\overrightarrow{x'x''} = \varepsilon_1\, C\, R\, h\, \tau_r + \varepsilon_2\, d\, \tau_t + \tau_\perp,$$

where $-1 < \varepsilon_1, \varepsilon_2 < 1$, and $\tau_\perp$ is perpendicular to both τ_r and τ_t. From (3.5) $d \le C\,R\sqrt{h}$ and so

$$|\tau_E \cdot \overrightarrow{x'x''}| \le |\varepsilon_1|\ |\tau_E \cdot \tau_r\, CRh| + |\varepsilon_2|\ |\tau_E \cdot \tau_t\, d| \le CRh + Cd^2/R \le CRh.$$

Note that $|\overrightarrow{C_0E_C}| \le C\,R$. It follows that

$$|\overrightarrow{C_0E_C} \cdot \overrightarrow{x'x''}| \le C\,R|\tau_E \cdot \overrightarrow{x'x''}| \le CR^2h.$$

Since $|\overrightarrow{x'x''}| \le d < R$, we have

$$\left|\left(E_C - 2a\frac{m'}{1-m_n}\right) \cdot \overrightarrow{x'x''}\right| \le CR^2h,$$

and then by the definition of R we obtain (3.6).

We are now ready to prove (a). Since $d_1 \le d \le d_2$, from (3.5) and (3.6), one obtains

$$|m' \cdot \overrightarrow{x_0'x'}| \le C\,h.$$

Since the ellipsoid E_0 has principal axes $C\lambda_1, \cdots, C\lambda_{n-1}$ in the coordinate directions $e_1, \cdots, e_{n-1}$, it follows from the last inequality that the i-th component m_i of m' must satisfy $|m_i| \le Ch/\lambda_i$.

We now prove (b). For $m' \in B_{\eta_0}(0)$ with small η_0, let $w = \mathcal{M}(m') = m' - \dfrac{1-\sqrt{1-|m'|^2}}{2a}E_C$. It is easy to verify that the Jacobian of $\mathcal{M}$ is close to 1 and that for m', $m_0' \in B_{\eta_0}(0)$ we have

$$(1 - C\eta_0)|m' - m_0'| \le |\mathcal{M}(m') - \mathcal{M}(m_0')| \le (1 + C\eta_0)|m' - m_0'|. \tag{3.7}$$

We claim that $\mathcal{M}$ is a 1-to-1 mapping from $B_{\eta_0}(0)$ onto $B_{\eta_0}(w_{\eta_0})$, where $w_\eta = -\dfrac{1-\sqrt{1-\eta^2}}{2a}E_C$ for $0 \le \eta \le \eta_0$. In fact, if $|m'| = \eta$, then $|w - w_\eta| = \eta$. It is easy to verify that $|w_\eta - w_{\eta_0}| \le C\eta_0(\eta_0 - \eta)$. Hence, $\mathcal{M}(B_{\eta_0}) \subset B_{\eta_0}(w_{\eta_0})$. On the other hand, given $w \ne 0$ with $|w - w_{\eta_0}| = \mu < \eta_0$, consider the continuous function $f(\eta) = |w - w_\eta|/\eta$ for $0 < \eta \le \eta_0$. Obviously, $\lim_{\eta\to 0^+} f(\eta) = \infty$ and $\lim_{\eta\to\eta_0} f(\eta) = \mu/\eta_0 < 1$. Therefore, there exists $0 < \eta < \eta_0$ such that $f(\eta) = 1$, which implies that $|w - w_\eta| = \eta$ and $w = \mathcal{M}(m')$ with $m' = w - w_\eta$. Thus, the claim is proved.

From (3.6), if $m = (m', m_n) \in \mathcal{N}_{\mathcal{A}}(\rho^{-1}(\mathcal{R}_{1/2}))$, then $w = \mathcal{M}(m') = (w_1, \cdots, w_{n-1})$ is in the dual ellipsoid E^* of the ellipsoid E given by $E^* = \{w : |w_i| \le Ch/\lambda_i,\ 1 \le i \le n-1\}$. Clearly, we have the following estimate

$$\begin{aligned}
|\mathcal{N}_{\mathcal{A}}(\rho^{-1}(\mathcal{R}_{1/2}))| &\le |\{(m', m_n) \in S^{n-1} : \sqrt{1-m_n} \le C\sqrt{h}/d \text{ and } \mathcal{M}(m') \in E^*\}| \\
&\le C|\{m' : |\mathcal{M}(m')| \le C\sqrt{h}/d \text{ and } \mathcal{M}(m') \in E^*\}| \\
&= C|\mathcal{M}^{-1}\{w : |w| \le C\sqrt{h}/d \text{ and } |w_i| \le Ch/\lambda_i,\ 1 \le i \le n-1\}| \\
&\le C\prod_{i=1}^{n-1} \min\left\{\frac{\sqrt{h}}{d}, \frac{h}{\lambda_i}\right\}.
\end{aligned}$$

This completes the proof of the theorem. □

A fundamental estimate for convex functions is the Alexandrov geometric inequality which asserts that if $u(x)$ is a convex function in a bounded convex domain $\Omega \subset \mathbb{R}^n$ such that $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$, then for $x_0 \in \Omega$

$$|u(x_0)|^n \leq C \operatorname{dist}(x_0, \partial\Omega) \operatorname{diam}(\Omega)^{n-1} |Du(\Omega)|;$$

see [Gut01, Lemma 1.4.2]. We extend this result to the setting of the reflector mapping in the following theorem.

THEOREM 3.4. *Let $\mathcal{A}$ be an admissible antenna satisfying* (2.1) *and let $P(e_n, a+h)$ with $h > 0$ small be a supporting paraboloid to $\mathcal{A}$. Denote by $\mathcal{R} = \mathcal{S}_{\mathcal{A}}(P(e_n, a))$ the portion of $\mathcal{A}$ bounded between $P(e_n, a+h)$ and $P(e_n, a)$, and let $\mathcal{R}'$ and E be defined as in Remark* 3.2. *Assume that E has center E_C and principal axes $\lambda_1, \cdots, \lambda_{n-1}$ in the coordinate directions $e_1, \cdots, e_{n-1}$. Denote by $\rho^{-1}(\mathcal{R})$ the preimage of $\mathcal{R}$ on S^{n-1}.*

(a) *Assume that $d_1 \leq d = \operatorname{diam}(E) \leq d_2$. Given $\delta > 0$ and $z' = (z_1, \cdots, z_{n-1}) \in \mathcal{R}'$ such that $z = (z', z_n) \in \mathcal{R} \cap P(e_n, a+h)$ with $K - \delta\lambda_1 \leq z_1 \leq K$, where $K = \sup_{x' \in \mathcal{R}'} x_1$, then there exists ε_0, independent of δ and z, such that*

$$\mathcal{F} = \{m \in S^{n-1} : \sqrt{1 - m_n} \leq \varepsilon_0 \sqrt{h}/d,$$
$$0 \leq -m_1 \leq \varepsilon_0 \frac{h}{\delta\,\lambda_1},\ |m_i| \leq \varepsilon_0 \frac{h}{\lambda_i},\ i = 2, \cdots, n-1\} \subset \mathcal{N}_{\mathcal{A}}(\rho^{-1}(\mathcal{R})).$$

In other words, if $m \in \mathcal{F}$, then $P(m, b)$ is a supporting paraboloid to $\mathcal{A}$ at some point on $\mathcal{R}$ for some $b > 0$.

(b) *Assume that $\sqrt{h}/d \leq C_0$. Let $\mathcal{B}$ be the linear transformation given by $\mathcal{B}(y_1, \cdots, y_{n-1}) = (\lambda_1 y_1, \cdots, \lambda_{n-1} y_{n-1})$ such that $E - E_C = \mathcal{B}B_1$, where B_1 is the unit ball. Given $\delta > 0$ and $z = (z', z_n) \in \mathcal{R} \cap P(e_n, a+h)$ with $z' = E_C + (\theta - \delta)\mathcal{B}y'$, $|y'| = 1$, $E_C + \theta\mathcal{B}y' \in \partial\mathcal{R}'$, and $\frac{1}{n-1} \leq \theta \leq 1$, then there exist a small $\varepsilon_0 > 0$, independent of δ and z, and $n-1$ orthonormal vectors $e_1^*, \cdots, e_{n-1}^*$ in $\mathbb{R}^{n-1}$ such that*

$$C\,|\{w \in \mathbb{R}^{n-1} : |w| \leq \varepsilon_0 \sqrt{h}/d,\ \mathcal{B}w \in E^*\}| \leq |\mathcal{N}_{\mathcal{A}}(\rho^{-1}(R))|,$$

where $E^* = \left\{ \sum_{i=1}^{n-1} w_i^* e_i^* : -\dfrac{\varepsilon_0 h}{3\delta} \leq w_1^* \leq 0,\ \sum_{i=2}^{n-1} (w_i^*)^2 \leq \left(\dfrac{\varepsilon_0 h}{3}\right)^2 \right\}$ *is a cylinder with circular base $B_{\varepsilon_0 h/3}$ and height $\dfrac{\varepsilon_0 h}{3\delta}$.*

Proof. Let $z \in \mathcal{R} \cap P(e_n, a+h)$. We have $1 - e_n \dfrac{z}{|z|} = \dfrac{2(a+h)}{|z|} \geq \text{const}$ by Remark 2.4 and (2.1). If $m \in S^{n-1}$ and $|m - e_n| \leq \varepsilon_0$ with ε_0 small, then

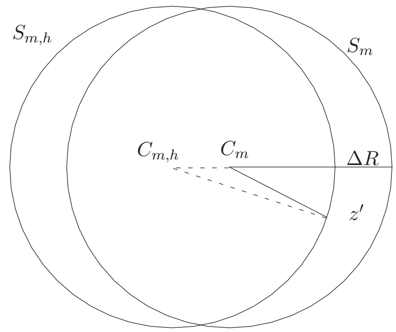

Figure 3: Theorem 3.4

$2b \triangleq |z|\left(1 - m\dfrac{z}{|z|}\right) \geq$ const and so $z \in P(m, b)$. Recall that E_C is the center of E and $\sqrt{h}/d \leq C_0$, and set

$$\mathcal{F}^* = \Bigg\{ m \in S^{n-1} : \sqrt{1 - m_n} \leq \varepsilon_0 \sqrt{h}/d,\ \sup_{x' \in \mathcal{R}'} \left(-m' + \frac{1 - m_n}{2a} E_C\right) \cdot \overrightarrow{z'x'} \leq \varepsilon_0 h \Bigg\}.$$

In order to prove (a) and (b), we first show that

$$\mathcal{F}^* \subset \mathcal{N}_{\mathcal{A}}(\rho^{-1}(\mathcal{R})). \tag{3.8}$$

To prove this, we will first claim that for $m \in \mathcal{F}^*$, the portion of $P(m, b)$ that contains z and is over $\mathcal{R}'$ is below $P(e_n, a)$, and second we will show that this implies that $P(m, b_0)$ (perhaps with b_0 different from b) is a supporting paraboloid to the whole antenna $\mathcal{A}$ at a point on $\mathcal{R}$.

To show the first claim, since z is below $P(e_n, a)$, it suffices to prove that

$$\mathcal{R}' \subset S_m, \tag{3.9}$$

where $S_m = S_{a,b,m}$ is the sphere from Lemma 2.1 which is the projection of the intersection of $P(e_n, a)$ and the bisector $\Pi[(e_n, a), (m, b)]$. As in the proof of Theorem 3.3, S_m has equation

$$S_m \equiv \left| x' - 2\,a \frac{m_\tau}{1 - m_n}\,\tau \right|^2 = \frac{8ab}{1 - m_n} = R_m^2 = R^2,$$

where $m_\tau = |m'|$ and $m' = m_\tau \tau$. In order to prove (3.9) we now show that

$$z' \text{ is inside } S_m \text{ and } \operatorname{dist}(z', S_m) \geq C\,R\,h, \tag{3.10}$$

and next construct a cylinder $\mathbb{C}$ defined by (3.11) so that $\mathcal{R}' \subset \mathbb{C} \subset S_m$. Indeed, since $z \in P(e_n, a + h) \cap P(m, b)$, z' must be on the sphere $S_{m,h} =$

$S_{a+h,b,m}$, the projection of the intersection of $P(e_n, a+h)$ and the bisector $\Pi[(e_n, a+h), (m, b)]$. Similarly to S_m, $S_{m,h}$ has equation

$$S_{m,h} \equiv \left| x' - 2\,(a+h)\frac{m_\tau}{1-m_n}\,\tau \right|^2 = \frac{8(a+h)b}{1-m_n} = R^2_{m,h}.$$

We claim that $|b-a| \le Cm_\tau + h \le C\varepsilon_0$. In fact, if $z = \rho(y)y$ for some $y \in S^{n-1}$, then

$$\rho(y) = \frac{2(a+h)}{1-e_n\cdot y} = \frac{2b}{1-m\cdot y}\,,$$

and consequently

$$\frac{b-a-h}{a+h} = \frac{(e_n - m)\cdot y}{1-e_n\cdot y}.$$

Therefore

$$|b-a-h| = \frac{a+h}{1-e_n\cdot y}\,|(e_n-m)\cdot y| \le \frac{1}{2}\,\rho(y)\,|e_n - m| \le Cm_\tau,$$

and the claim follows.

Let C_m and $C_{m,h}$ be the centers of S_m and $S_{m,h}$ respectively. By the law of cosines

$$1+\tau\cdot\frac{\overrightarrow{C_{m,h}z'}}{|\overrightarrow{C_{m,h}z'}|} = 1 - \frac{\overrightarrow{C_{m,h}z'}}{|\overrightarrow{C_{m,h}z'}|}\cdot\frac{\overrightarrow{C_{m,h}O}}{|\overrightarrow{C_{m,h}O}|} \le \frac{|Oz'|^2}{2|\overrightarrow{C_{m,h}z'}|\cdot|\overrightarrow{C_{m,h}O}|} \le \frac{C}{R^2},$$

where O is the origin in $\mathbb{R}^{n-1}$. This means that the angle between $-\tau$ and $\overrightarrow{C_{m,h}z'}$ is less than C/R. We now estimate $|\overrightarrow{C_m z'}|$ to locate the position of z' inside S_m. Again by using the inner product, we have

$$\begin{aligned}
|\overrightarrow{C_m z'}|^2 &= |\overrightarrow{C_{m,h}z'}|^2 + |\overrightarrow{C_{m,h}C_m}|^2 - 2\overrightarrow{C_{m,h}z'}\,\overrightarrow{C_{m,h}C_m}\\
&= \left(R_{m,h} - 2h\frac{m_\tau}{1-m_n}\right)^2 + 2R_{m,h}\cdot 2h\frac{m_\tau}{1-m_n}\left[1 - \frac{\overrightarrow{C_{m,h}z'}}{|\overrightarrow{C_{m,h}z'}|}(-\tau)\right]\\
&\le \left(R_{m,h} - 2h\frac{m_\tau}{1-m_n}\right)^2 + CR\cdot hR\cdot\frac{C}{R^2}\\
&\le \left(R_{m,h} - 2h\frac{m_\tau}{1-m_n}\right)^2 + C\,h.
\end{aligned}$$

Since $R_{m,h} - 2h\frac{m_\tau}{1-m_n} \approx R$, it follows that

$$|\overrightarrow{C_m z'}| \le \left(R_{m,h} - 2h\frac{m_\tau}{1-m_n}\right) + C\frac{h}{R}.$$

On the other hand

$$\begin{aligned}\Delta R &\triangleq R - \left(R_{m,h} - 2h\frac{m_\tau}{1-m_n}\right) \\ &= 2h\frac{m_\tau}{1-m_n} + \sqrt{\frac{8ab}{1-m_n}} - \sqrt{\frac{8(a+h)b}{1-m_n}} \\ &= \frac{2h\sqrt{1+m_n}}{\sqrt{1-m_n}} - \frac{\sqrt{8b}}{\sqrt{1-m_n}}\frac{h}{\sqrt{a+h}+\sqrt{a}} \\ &\geq \frac{h}{\sqrt{1-m_n}}\left[2\sqrt{1+m_n} - \sqrt{2}\sqrt{\frac{b}{a}}\right].\end{aligned}$$

Since $|b-a| \leq C\varepsilon_0$ and m_n is close to 1, we obtain that $\Delta R \geq \dfrac{C\,h}{\sqrt{1-m_n}} = CRh$. Therefore

$$\begin{aligned}R - |\overrightarrow{C_m z'}| &\geq R - \left(R_{m,h} - 2h\frac{m_\tau}{1-m_n}\right) - C\frac{h}{R} = \Delta R - C\frac{h}{R} \\ &\geq C\,R\,h.\end{aligned}$$

So (3.10) is proved.

Write $\overrightarrow{C_m z'} = |\overrightarrow{C_m z'}|\tau_r$, i.e., τ_r is the radial direction at z'. We obtain that the cylinder

(3.11)
$$\mathbb{C} = \{z' + k_r\tau_r + k_t\tau_t : -R/2 \leq k_r \leq CRh,\, |k_t| \leq CR\sqrt{h},\, \tau_t \perp \tau_r,\, |\tau_t| = 1\}$$

is contained inside the sphere S_m for an appropriate choice of C.

We next prove that if $m \in \mathcal{F}^*$, then $\mathcal{R}' \subset \mathbb{C}$ which will complete the proof of (3.9). Write $\overrightarrow{C_m E_C} = |\overrightarrow{C_m E_C}|\tau_E$. Then the angle between τ_r and τ_E is small. Indeed, by the law of cosines

$$1 - \tau_r\tau_E \leq \frac{|\overrightarrow{E_C z'}|^2}{2|\overrightarrow{C_m z'}|\cdot|\overrightarrow{C_m E_C}|} \leq \frac{C\,d^2}{R^2}.$$

Write $\tau_r = k_E\tau_E + k_*\tau_*$ such that $k_* \geq 0$, $\tau_* \perp \tau_E$, and $|\tau_*| = 1$. Therefore, $\dfrac{1}{2} \leq k_E \leq 1$ and $0 \leq k_* \leq C\dfrac{d}{R}$. Since $m \in \mathcal{F}^*$ and $R = C/\sqrt{1-m_n}$, $|\overrightarrow{z'x'}\cdot\tau_t| \leq |\overrightarrow{z'x'}| \leq d \leq \varepsilon_0 CR\sqrt{h}$ for $x' \in \mathcal{R}'$, and we have the following estimate

$$\begin{aligned}\overrightarrow{z'x'}\cdot\tau_r = k_E\,\overrightarrow{z'x'}\cdot\tau_E + k_*\,\overrightarrow{z'x'}\cdot\tau_* &\leq k_E\frac{\overrightarrow{z'x'}\cdot\left(E_C - 2a\frac{m'}{1-m_n}\right)}{|\overrightarrow{C_m E_C}|} + C\frac{d}{R}d \\ &\leq k_E\frac{\frac{2a}{1-m_n}\,\varepsilon_0 h}{|\overrightarrow{C_m E_C}|} + C\frac{d^2}{R} \leq C\varepsilon_0\,R\,h + C\varepsilon_0^2 R\,h = C\varepsilon_0 R\,h,\end{aligned}$$

for all $x' \in \mathcal{R}'$, and therefore $\mathcal{R}' \subset \mathbb{C}$ and so (3.9) follows. As a result, the portion of $P(m,b)$ over $\mathcal{R}'$ passing through z is strictly below (or outside) $P(e_n,a)$. Furthermore, the portion of $P(e_n,a)$ over $\mathcal{R}'$ is strictly contained in $P(m,b)$. Geometrically, if we drag $P(m,b)$ downward (i.e., having b increase), then we can get a supporting paraboloid $P(m,b_0)$. In fact, if $x=\rho(y)y \in \mathcal{R}$ with $|y|=1$, then $x \in P(e_n, a+\varepsilon h)$ for some $0 \le \varepsilon \le 1$ and $1-e_n y = 2(a+\varepsilon h)/\rho(y) \ge \text{const}$. Thus, $x \in P(m,b_x)$ where $2b_x \triangleq \rho(y)(1-my) \ge \text{const}$. Let $b_0 = \sup\{b_x : x \in \mathcal{R}\}$. We want to prove that $P(m,b_0)$ is a supporting paraboloid to $\mathcal{A}$ at a point in the interior of $\mathcal{R}$. Choose $z_k \in \mathcal{R}$ such that $b_{z_k} \to b_0$. Without loss of generality, assume that $z_k \to z_0$. Let $z_k = \rho(y_k)y_k$, $|y_k|=1$, and $z_0 = \rho(y_0)y_0$, $|y_0|=1$. By taking the limit as $k \to \infty$ in the equation $\rho(y_k)(1-my_k) = 2b_{z_k}$, we get that $z_0 \in P(m,b_0)$. On the other hand, every $x \in \mathcal{R}$ must be on some $P(m,b_x)$ which lies inside $P(m,b_0)$ since $b_x \le b_0$. Hence, $\mathcal{R}$ is inside $P(m,b_0)$ and touches $P(m,b_0)$ at z_0. It follows that $P(m,b_0)$ is a supporting paraboloid to $\mathcal{R}$ and $\partial\mathcal{R}$ is strictly inside $P(m,b_0)$. To show that $P(m,b_0)$ is a supporting paraboloid to $\mathcal{A}$, it is enough to show that $\mathcal{A}\setminus\mathcal{R}$ is also contained inside $P(m,b_0)$. Indeed, suppose by contradiction that there exists $x \in \mathcal{A}\setminus\mathcal{R}$ lying outside $P(m,b_0)$. Then x, z_0 lie on or outside $P(m,b_0)$. By Lemma 3.1, there exists a curve $\mathcal{C}$ on $\mathcal{A}$ connecting z_0 and x and lying on or outside $P(m,b_0)$. Then $\mathcal{C}$ must cross the boundary of $\mathcal{R}$ which is strictly contained inside $P(m,b_0)$, a contradiction. Thus, the proof of (3.8) is complete.

Now to the proof of Part (a). Given $x' \in \mathcal{R}'$, write $\overrightarrow{z'x'} = x'-z' = \varepsilon_1\lambda_1 e_1 + \sum_{i=2}^{n-1}\varepsilon_i\lambda_i e_i$, where $-2 \le \varepsilon_1 \le \delta$ and $-2 \le \varepsilon_i \le 2$ for $i=2,\cdots,n-1$. If $m \in \mathcal{F}$ and since $d_1 \le d \le d_2$, then one obtains

$$\begin{aligned}\overrightarrow{z'x'}\left(\frac{1-m_n}{2a}E_C - m'\right) &\le C\varepsilon_0^2 h - \varepsilon_1 m_1\lambda_1 - \sum_{i=2}^{n-1}\varepsilon_i m_i\lambda_i \\ &\le C\varepsilon_0^2 h + \delta\frac{\varepsilon_0 h}{\delta\lambda_1}\lambda_1 + 2(n-2)\varepsilon_0 h \\ &= C\,\varepsilon_0 h.\end{aligned}$$

Choosing ε_0 in $\mathcal{F}$ sufficiently small we get that $\mathcal{F} \subset \mathcal{F}^*$ ($\mathcal{F}^*$ is now defined with $C\varepsilon_0$ instead of ε_0) and (a) follows from (3.8).

To prove (b), and as in the proof of Theorem 3.3, we consider the mapping $w = \mathcal{M}(m') = m' - \dfrac{1-\sqrt{1-|m'|^2}}{2a}E_C$. Let

$$\operatorname{Proj}\mathcal{F}^* = \{m' : \exists\, m_n \text{ such that } (m', m_n) \in \mathcal{F}^*\}$$

be the projection on $\mathbb{R}^{n-1}$. Since ε_0 is small, it follows from (3.7) that

$$\begin{aligned} &\{m' : |\mathcal{M}(m')| \le \frac{\varepsilon_0}{2}\frac{\sqrt{h}}{d},\ \sup_{x'\in\mathcal{R}'}[-\mathcal{M}(m')]\, z'x' \le \varepsilon_0 h\} \\ &\quad\subset \{m' : |m'| \le \varepsilon_0\sqrt{h}/d,\ \sup_{x'\in\mathcal{R}'}[-\mathcal{M}(m')]\, z'x' \le \varepsilon_0 h\} \\ &\quad\subset \operatorname{Proj}\mathcal{F}^*. \end{aligned} \tag{3.12}$$

We claim that

$$\{m' : \mathcal{B}\mathcal{M}(m') \in E^*\} \subset \{m' : \sup_{x'\in\mathcal{R}'}[-\mathcal{M}(m')]\, z'x' \le \varepsilon_0 h\}.$$

Let $\mathcal{R}^* = \mathcal{B}^{-1}(\mathcal{R}' - E_C)$. Obviously, $B_{\frac{1}{n-1}} \subset \mathcal{R}^* \subset B_1$. By the assumptions, $E_C + \theta\mathcal{B}y' \in \partial\mathcal{R}'$ and hence $\theta y' \in \partial\mathcal{R}^*$. Let $z^* \triangleq (\theta-\delta)y' = \mathcal{B}^{-1}(z' - E_C)$, and let $y^* \in \partial\mathcal{R}^*$ be such that $\operatorname{dist}(z^*, \partial\mathcal{R}^*) = |y^* - z^*|$. Let $e_1^* = \dfrac{y^*-z^*}{|y^*-z^*|}$ and choose $\{e_i^*\}_{i=2}^{n-1}$ such that $\{e_i^*\}_{i=1}^{n-1}$ is a set of orthonormal vectors. Clearly, e_1^* is normal to $\partial\mathcal{R}^*$ at y^* and

$$\mathcal{R}^* \subset \{z^* + \sum_{i=1}^{n-1} u_i e_i^* : -2 \le u_1 \le \delta, |u_i| \le 2,\ i = 2, \cdots, n-1\}.$$

Therefore, for $\mathcal{B}w \in E^*$, it is easy to verify

$$\sup_{x'\in\mathcal{R}'}(-w)\cdot z'x' = \sup_{u\in\mathcal{R}^*}[-\mathcal{B}w]\cdot(u - z^*) \le \varepsilon_0 h,$$

and the claim follows. Therefore from (3.12) we get

$$\{m' : |\mathcal{M}(m')| \le \frac{\varepsilon_0}{2}\frac{\sqrt{h}}{d},\ \mathcal{B}\mathcal{M}(m') \in E^*\} \subset \operatorname{Proj}\mathcal{F}^*.$$

Since the Jacobian of $\mathcal{M}$ is close to one, the conclusion in part (b) follows from (3.8). □

3.3. *Estimates in case the diameter of E is small.* Theorem 3.3 and Theorem 3.4 extend the gradient estimate and Alexandrov estimate for convex functions to reflector antennas in the case $\sqrt{h}/d \le \eta_0$. These two theorems are sufficient for the discussion of strict reflector antennas in Section 4. However, to get complete extension of the estimates, we also need to prove the following theorem addressing the case $\sqrt{h}/d \ge \eta_0$.

THEOREM 3.5. *Let $\mathcal{A}$ be an admissible antenna satisfying* (2.1) *and* (3.1), *and let $P(e_n, a+h)$ be a supporting paraboloid to $\mathcal{A}$ for small $h > 0$. Let $\mathcal{R} = \mathcal{S}_{\mathcal{A}}(P(e_n, a))$ and let $\mathcal{R}'$ and E be defined as in Remark* 3.2, *let E_C be the center of E, and $d = \operatorname{diam}(E)$. Assume that $\dfrac{\sqrt{h}}{d} \ge \eta_0 > 0$.*

(a) *There exists $C > 0$ such that $C^{-1}d \le \lambda_i \le C\,d$, for $i = 1, \cdots, n-1$, and $\eta_0 \le \dfrac{\sqrt{h}}{d} \le C$.*

(b) *Let $\mathcal{B}$ be the linear transformation given by*

$$\mathcal{B}(y_1, \cdots, y_{n-1}) = (\lambda_1 y_1, \cdots, \lambda_{n-1} y_{n-1})$$

and be such that $E - E_C = \mathcal{B}B_1$. Given $\delta > 0$ and $z = (z', z_n) \in \mathcal{R} \cap P(e_n, a+h)$ such that $z' = E_C + (1-\delta)\mathcal{B}y'$ with $\frac{1}{n-1} \leq |y'| \leq 1$, and $E_C + \mathcal{B}y' \in \partial\mathcal{R}'$, there exists a small $\varepsilon_0 > 0$ such that

$$C\varepsilon_0^{n-1} \min\left\{\frac{1}{\sqrt{h}}, \frac{1}{\delta}\right\} \left(\sqrt{h}\right)^{n-1} \leq |\mathcal{N}_{\mathcal{A}}(\rho^{-1}(\mathcal{R}))|,$$

where $\rho^{-1}(\mathcal{R})$ is the preimage of $\mathcal{R}$ on S^{n-1}.

(c) *Let $\mathcal{R}_{1/2}$ be the lower portion of $\mathcal{R}$ whose projection onto $\mathbb{R}^{n-1}$ is $\frac{1}{2(n-1)}E$ and $\rho^{-1}(\mathcal{R}_{1/2})$ be its preimage on S^{n-1}. Then*

$$\mathcal{N}_{\mathcal{A}}(\rho^{-1}(\mathcal{R}_{1/2})) \subset \{(m', m_n) \in S^{n-1} : \sqrt{1-m_n} \leq C\sqrt{h}\},$$

where C depends only on the structural constants and η_0.

Proof. We first prove part (a). Let $z = (z', z_n) \in \mathcal{R} \cap P(e_n, a+h)$. We remark that to prove (3.10), it suffices to assume that $|m - e_n| \leq \varepsilon_0$ with ε_0 small, and therefore under this assumption one can conclude as in Theorem 3.4 that the cylinder

$$\mathbb{C} = \{z' + k_r\tau_r + k_t\tau_t : -R/2 \leq k_r \leq CRh,\ |k_t| \leq CR\sqrt{h},\ \tau_t \perp \tau_r,\ |\tau_t| = 1\}$$

is contained strictly inside the sphere S_m, where the symbols have the same meaning as in that theorem. If on the other hand $\sqrt{1-m_n} \leq \varepsilon_0 h/d$, then $d \leq \varepsilon_0 CRh$ where R is the radius of S_m and $R = C/\sqrt{1-m_n}$ and consequently $\mathcal{R}' \subset B_d(z') \subset \mathbb{C}$. Therefore, if $\sqrt{1-m_n} \leq \min\{\varepsilon_0, \varepsilon_0 h/d\}$, then $\mathcal{R}' \subset S_m$. Using the technique of dragging the paraboloid as in the proof of Theorem 3.4, we then obtain that $m \in \mathcal{N}_{\mathcal{A}}(\rho^{-1}(\mathcal{R}))$; that is,

$$\{m \in S^{n-1} : \sqrt{1-m_n} \leq \min\{\varepsilon_0, \varepsilon_0 h/d\}\} \subset \mathcal{N}_{\mathcal{A}}(\rho^{-1}(\mathcal{R})).$$

Let $D = \rho^{-1}(\mathcal{R})$ and $P(e_n, a)|_D$ be the restriction of $P(e_n, a)$ over D, i.e., the portion of $P(e_n, a)$ contained in $\mathcal{A}$. Obviously, $|D| \leq C\,|P(e_n, a)|_D| \leq C\,|\mathcal{R}'|$. By (3.1), we obtain

$$[\min\{\varepsilon_0, \varepsilon_0 h/d\}]^{n-1} \leq C\,|D| \leq C\,|\mathcal{R}'| \leq C\lambda_1 \cdots \lambda_{n-1}. \tag{3.13}$$

We claim that if h is small enough, then $h/d \leq 1$. Otherwise, if $h/d > 1$, then from (3.13), $\varepsilon_0^{n-1} \leq C\lambda_1 \cdots \lambda_{n-1}$. This implies that $\lambda_i \geq C\varepsilon_0^{n-1}$ for $i = 1, \cdots, n-1$, and therefore $\dfrac{\sqrt{h}}{d} \leq \dfrac{\sqrt{h}}{C\varepsilon_0^{n-1}} < \eta_0$ for small h, a contradiction. Thus, the claim is proved. Therefore from (3.13), $\left(\varepsilon_0 \dfrac{h}{d}\right)^{n-1} \leq C\lambda_1 \cdots \lambda_{n-1}$, and so

$$(\varepsilon_0\eta_0^2)^{n-1} \le \varepsilon_0^{n-1}\left(\frac{h}{d^2}\right)^{n-1} \le C\frac{\lambda_1\cdots\lambda_{n-1}}{d^{n-1}} \le C\frac{\lambda_i}{d} \le C,$$

for $i = 1, \cdots, n-1$. This completes the proof of part (a).

Now prove part (b). By part (a), $\eta_0 \le \sqrt{h}/d \le C$. By Theorem 3.4 (b)

$$C\,|\{w \in \mathbb{R}^{n-1} : |w| \le \varepsilon_0\eta_0, \mathcal{B}w \in E^*\}| \le |\mathcal{N}_{\mathcal{A}}(\rho^{-1}(\mathcal{R}))|,$$

where E^* is a cylinder with circular base $B_{\varepsilon_0 h/3}$ and height $\dfrac{\varepsilon_0 h}{3\delta}$. By part (a), $|\mathcal{B}w| \approx Cd|w|$. Therefore

$$\begin{aligned} C\,|\mathcal{B}^{-1}(B_{C_0\varepsilon_0\eta_0 d} \cap E^*)| \le& C|\{w \in \mathbb{R}^{n-1} : |\mathcal{B}w| \le C_0\varepsilon_0\eta_0 d, \mathcal{B}w \in E^*\}| \\ \le& |\mathcal{N}_{\mathcal{A}}(\rho^{-1}(\mathcal{R}))|. \end{aligned}$$

Since $\sqrt{h}/C \le d \le \sqrt{h}/\eta_0$, it is easy to verify that

$$\begin{aligned} |\mathcal{N}_{\mathcal{A}}(\rho^{-1}(\mathcal{R}))| &\ge \frac{C}{d^{n-1}}\min\left\{C_0\varepsilon_0\eta_0 d, \frac{\varepsilon_0 h}{3\delta}\right\}\left[\min\left\{C_0\varepsilon_0\eta_0 d, \frac{\varepsilon_0 h}{3}\right\}\right]^{n-2} \\ &\ge C(\varepsilon_0\eta_0)^{n-1}\min\left\{\frac{1}{\sqrt{h}}, \frac{1}{\delta}\right\}\left(\sqrt{h}\right)^{n-1}. \end{aligned}$$

This completes the proof of part (b).

To prove part (c), let $z = (z', z_n) \in \mathcal{R}_{1/2}$ and $P(m, b)$ be a supporting paraboloid at z. As in the proof of (3.5) in Theorem 3.3, there exists an ellipsoid $E_0 \subset \mathcal{R}'$ whose axes are comparable and parallel to those of E such that E_0 is contained in a cylinder $\mathbb{C}$ whose height is $C\,R\,h$ and whose base is an $(n-2)$-dimensional ball with radius $CR\sqrt{h}$ and center z', where $R = C/\sqrt{1-m_n}$. By part (a), it follows that $B_{\sigma_0 d} \subset \mathbb{C}$ for some small σ_0. Therefore, $\sigma_0 d \le CRh = Ch/\sqrt{1-m_n}$. Since $d \approx C\sqrt{h}$, we obtain that $\sqrt{1-m_n} \le C\sqrt{h}$. The proof of the theorem is finished. □

4. Strict antennas

In this section, we use the estimates established in Section 3 to show that a reflector antenna satisfying (2.1) and (3.1) must be a strict reflector antenna.

Definition 4.1 (Strict antenna). An admissible antenna $\mathcal{A}$ is a *strict antenna* if every supporting paraboloid of $\mathcal{A}$ touches $\mathcal{A}$ at only one point.

The following result is concerned with strict antenna.

THEOREM 4.2. *If $\mathcal{A}$ is an admissible antenna satisfying* (2.1) *and* (3.1), *then $\mathcal{A}$ is a strict antenna, and consequently, the map $\mathcal{N}_{\mathcal{A}}$ is injective.*

Proof. Let $P(e_n, a_1)$ be a supporting paraboloid to $\mathcal{A}$. We need to show that $P(e_n, a_1) \cap \mathcal{A}$ is a single point set. By Lemma 3.1 (b), the projection Δ on $\mathbb{R}^{n-1}$ of $P(e_n, a_1) \cap \mathcal{A}$ is a convex set. Suppose by contradiction that Δ contains at least two points. Then $\operatorname{diam}(\Delta) = \textit{constant} > 0$. For h sufficiently small, let $\mathcal{R}_h$ be the portion of $\mathcal{A}$ cut by $P(e_n, a_1 - h)$, $\mathcal{R}_0$ the portion of $\mathcal{A}$ cut by $P(e_n, a_1)$ and relabel $a = a_1 - h$, $a + h = a_1$.

We claim that $\mathcal{R}_h$ converges to $\mathcal{R}_0$ in the Hausdorff metric as $h \to 0$. Indeed, suppose by contradiction that there exist $\delta_0 > 0$ and $z_h \in \mathcal{R}_h$ such that $\operatorname{dist}(z_h, \mathcal{R}_0) \geq \delta_0$. By compactness, passing through a subsequence $z_h \to z_0 \in \mathcal{R}_0$ and $|z_h - z_0| \geq \delta_0$, we obtain a contradiction.

Let $\mathcal{R}'_h$ be the projection of $\mathcal{R}_h$ on $\mathbb{R}^{n-1}$. Then by the claim, $\mathcal{R}'_h \to \Delta$ in the Hausdorff metric as $h \to 0$. Let E_h be the John ellipsoid for the set $\mathcal{R}'_h$ and let $\lambda_1(h)$ be the longest axis of E_h. Then $\lambda_1(h) \approx C \approx \operatorname{diam}(\Delta)$ and there exists $z_h \in \Delta$ such that $K - \delta_h \lambda_1(h) \leq (z_h)_1 \leq K$, where $K = \sup_{z \in \mathcal{R}'_h} z_1$. Notice that $\delta_h \to 0$ as $h \to 0$. We now apply Theorems 3.3 and 3.4 to get a contradiction. Let $\hat{\mathcal{R}}_h$ and $(\hat{\mathcal{R}}_h)_{1/2}$ be the lower portions of $\mathcal{R}_h$ defined over $\mathcal{R}'_h$ and $\frac{1}{2(n-1)} E_h$, respectively, and let D_h and $(D_h)_{1/2}$ be the preimages on S^{n-1} of $\hat{\mathcal{R}}_h$ and $(\hat{\mathcal{R}}_h)_{1/2}$, respectively. We want to show that $|D_h| \approx |\mathcal{R}'_h|$. Given $y = \rho(x)\, x \in \mathcal{A}$ with $x \in S^{n-1}$, let $P(e, b)$ be a supporting paraboloid to $\mathcal{A}$ at y. Let $\overrightarrow{n}$ be the inner normal of $P(e, b)$ and $\mathcal{A}$ at y. Then by Snell's law, $n \cdot (-x) = n \cdot e$ and $n \cdot (-x) \geq \text{const} > 0$. It follows that $|D_h| \approx |\hat{\mathcal{R}}_h|$. Similarly, $|D_h| \approx |P(e_n, a)|_{D_h}|$, where $P(e_n, a)|_{D_h}$ is the restriction of $P(e_n, a)$ on D_h. Obviously,

$$|D_h| \leq C\, |P(e_n, a)|_{D_h}| \leq C\, |\mathcal{R}'_h| \leq C\, |\hat{\mathcal{R}}_h| \leq C\, |D_h|.$$

This proves that $|D_h| \approx |\mathcal{R}'_h|$.

Since $|(D_h)_{1/2}| \approx |(\hat{\mathcal{R}}_h)_{1/2}|$, to show that

$$|(D_h)_{1/2}| \approx |\tfrac{1}{2(n-1)} E_h|, \tag{4.1}$$

it suffices to prove that $(\hat{\mathcal{R}}_h)_{1/2}$ is a Lipschitz graph. For $y = \rho(x)\, x \in (\hat{\mathcal{R}}_h)_{1/2}$, let $P(m, b)$ be a supporting paraboloid to $\mathcal{A}$ at y, and $\overrightarrow{n}$ be the inner normal to $P(m, b)$ and $(\hat{\mathcal{R}}_h)_{1/2}$ at y. By Theorem 3.3(a), $|e_n - m| \leq C\sqrt{h}$. Since $P(m, b)$ is smooth, $m \cdot \overrightarrow{n} \geq \text{const} > 0$. This implies that $e_n \cdot \overrightarrow{n} \geq \text{const} > 0$. Therefore $(\hat{\mathcal{R}}_h)_{1/2}$ is a Lipschitz graph and so (4.1) holds.

From Theorem 3.3(a) and (3.1) we get

$$C\, |E_h| \leq |\mathcal{N}_{\mathcal{A}}((D_h)_{1/2})| \leq \min\left\{\frac{C\, h}{\lambda_1}, \frac{C\sqrt{h}}{\operatorname{diam}(E_h)}\right\} \prod_{i=2}^{n-1} \min\left\{\frac{C\, h}{\lambda_i}, \frac{C\sqrt{h}}{\operatorname{diam}(E_h)}\right\}.$$

On the other hand, from Theorem 3.4(a) and (3.1) we have

$$|E_h| \geq |\mathcal{R}'_h| \geq C\,|\mathcal{N}_{\mathcal{A}}(D_h)| \geq C \min\left\{\frac{\varepsilon_0\, h}{\delta_h\, \lambda_1}, \frac{\varepsilon_0\sqrt{h}}{\operatorname{diam}(E_h)}\right\} \prod_{i=2}^{n-1} \min\left\{\frac{\varepsilon_0\, h}{\lambda_i}, \frac{\varepsilon_0\sqrt{h}}{\operatorname{diam}(E_h)}\right\}.$$

Therefore,

$$\varepsilon_0^{n-1} \min\left\{\frac{h}{\delta_h\,\lambda_1}, \frac{\sqrt{h}}{\operatorname{diam}(E_h)}\right\} \leq C \min\left\{\frac{h}{\lambda_1}, \frac{\sqrt{h}}{\operatorname{diam}(E_h)}\right\}.$$

Since $\lambda_1 \approx \operatorname{diam}(E_h) \approx$ const, we obtain for any sufficiently small $h > 0$

$$\varepsilon_0^{n-1} \min\left\{\frac{h}{\delta_h}, \sqrt{h}\right\} \leq C\, h,$$

which gives a contradiction. □

Remark 4.3. We notice that if $\mathcal{A}_k = \{x\rho_k(x) : x \in S^{n-1}\}$ is a sequence of admissible antennas satisfying (2.1), then $\mathcal{A}_k$ converges to the antenna $\mathcal{A} = \{x\rho(x) : x \in S^{n-1}\}$ in the Hausdorff metric if and only if ρ_k converges to ρ uniformly on S^{n-1}.

LEMMA 4.4. *Let* $\mathcal{A}_j = \{x\rho_j(x) : x \in S^{n-1}\}$, $j \geq 1$, *be admissible antennas satisfying* (2.1). *Assume that* ρ_j *converges to* ρ *uniformly on* S^{n-1}. *Then*

(a) $\limsup_{j\to\infty} |\mathcal{N}_{\mathcal{A}_j}(K)| \leq |\mathcal{N}_{\mathcal{A}}(K)|$, *for any compact set* $K \subset S^{n-1}$;

(b) $\liminf_{j\to\infty} |\mathcal{N}_{\mathcal{A}_j}(O)| \geq |\mathcal{N}_{\mathcal{A}}(O)|$, *for any open set* $O \subset S^{n-1}$.

Proof. Part (a) is easy to prove by definition. For completeness, we prove part (b). Let $K \subset O$ be compact and $E^* = \cup_{x_1 \neq x_2}[\mathcal{N}_{\mathcal{A}}(x_1) \cap \mathcal{N}_{\mathcal{A}}(x_2)]$. Then $|E^*| = 0$. Let $m_0 \in \mathcal{N}_{\mathcal{A}}(K) \setminus E^*$. There exist $x_0 \in K$ and $a > 0$ such that $P(m_0, a)$ is a supporting paraboloid to $\mathcal{A}$ at $x_0\rho(x_0)$. To finish the proof, it suffices to show that $m_0 \in \mathcal{N}_{\mathcal{A}_j}(O)$ for sufficiently large j. Since $m_0 \notin E^*$, $\lim_{h\to 0} \operatorname{diam}(S_{\mathcal{A}}(P(m_0, a-h))) = 0$. By the continuity of the mapping $\frac{y}{|y|}$, $\lim_{h\to 0} \operatorname{diam}(D_h) = 0$, where $D_h = \rho^{-1}(S_{\mathcal{A}}(P(m_0, a-h)))$ is the radial projection on S^{n-1} of $S_{\mathcal{A}}(P(m_0, a-h))$. Choose $D_h \subset\subset O$. Let $\varepsilon > 0$ and choose j_0 large. If $j \geq j_0$, then for $x \in S^{n-1} \setminus D_h$,

$$\rho_j(x) \leq (1+\varepsilon)\rho(x) \leq (1+\varepsilon)\frac{a-h}{1 - m_0 \cdot x},$$

and

$$\rho_j(x_0) \geq (1-\varepsilon)\rho(x_0) = (1-\varepsilon)\frac{a}{1 - m_0 \cdot x_0}.$$

Let $b_0 = (1+\varepsilon)(a-h) > 0$ and choose ε small enough such that $\delta = (1-\varepsilon)a - b_0 > 0$. Then $\mathcal{A}_j$ is inside $P(m_0, b_0)$ along directions in $S^{n-1} \setminus D_h$

and $\mathcal{A}_j$ is outside $P(m_0, b_0 + \delta)$ in the direction x_0. For $x \in D_h$, since $a - h \le \rho(x)(1 - m_0 \cdot x) \le a$, $2b_x \triangleq \rho_j(x)(1 - m_0 \cdot x) > 0$ for $j \ge j_0$. This means that $x\rho_j(x) \in P(m_0, b_x)$. Let $b = \sup\{b_x : x \in D_h \text{ and } x\rho_j(x) \in P(m_0, b_x)\}$. Without loss of generality, assume that $b = b_{x_1}$ where $x_1 \in D_h$. Obviously, $b \ge b_0 + \delta$. Therefore, $P(m_0, b)$ is a supporting paraboloid to $\mathcal{A}_j$ at $x_1\rho_j(x_1) \in O$. This completes the proof of the lemma. □

Corollary 4.5. *The class of admissible antennas satisfying* (2.1) *and* (3.1) *is compact with respect to the Hausdorff metric.*

Proof. By Remark 4.3 and Lemma 4.4, to prove the corollary it suffices to estimate uniformly the Lipschitz constant of the radial function defining the antennas. Let $\mathcal{A}$ be an antenna parametrized by $\rho(x)$ such that (2.1) and (3.1) hold. Let $x_0,\, x_1 \in S^{n-1}$ and $|x_0 - x_1| \le \varepsilon_0$. Let $P(m_0, a_0)$ be a supporting paraboloid of $\mathcal{A}$ at $x_0\rho(x_0)$. Obviously

$$\rho(x_1) - \rho(x_0) \le \frac{2a_0}{1 - m_0 x_0} \frac{m_0(x_1 - x_0)}{1 - m_0 x_1}.$$

Since $1 - m_0 x_0 = 2a_0/\rho(x_0) \ge \text{const}$, then $1 - m_0 x_1 \ge \text{const}$, if ε_0 is small. We conclude that $\rho(x_1) - \rho(x_0) \le C|x_0 - x_1|$. The corollary is proved. □

As a corollary of Theorem 4.2 and Corollary 4.5, we have the following result on the diameter of sections.

Corollary 4.6. *Let $\mathcal{A}$ be an admissible antenna satisfying* (2.1) *and* (3.1). *Then there exists an increasing function $\sigma(h)$ depending only on r_1, r_2, λ, Λ, and n with $\lim_{h \to 0^+} \sigma(h) = 0$ such that* $\operatorname{diam}(\mathcal{S}_{\mathcal{A}}(P(m, b - h))) \le \sigma(h)$ *for any supporting paraboloid $P(m, b)$ of $\mathcal{A}$.*

5. Legendre transform

Our purpose in this section is to discuss some properties of the Legendre transform (see Definition 5.1) of weak solutions to the reflector antenna problem, a notion introduced in [GW98].

Definition 5.1 (Legendre transform). Given an admissible antenna $\mathcal{A} = \{x\rho(x) : x \in S^{n-1}\}$, the *Legendre transform* of $\mathcal{A}$, denoted by $\mathcal{A}^*$, is defined by $\mathcal{A}^* = \{m\rho^*(m) : m \in S^{n-1}\}$, where

$$\rho^*(m) = \inf_{x \in S^{n-1}, x \ne m} \frac{1}{\rho(x)(1 - m \cdot x)} = \frac{1}{\sup_{x \in S^{n-1}} [\rho(x)(1 - m \cdot x)]}.$$

Lemma 5.2. *If $\mathcal{A}$ is an admissible antenna satisfying* (2.1), *then its Legendre transform $\mathcal{A}^*$ is also an admissible antenna and satisfies* $\dfrac{1}{2r_2} \le$

$\rho^*(m) \leq \dfrac{1}{2r_1}$ *for any* $m \in S^{n-1}$*. Moreover, if* $m_0 \in \mathcal{N}_{\mathcal{A}}(x_0)$*, then* $x_0 \in \mathcal{N}_{\mathcal{A}^*}(m_0)$.

Proof. The proof is similar to that of Lemma 4.1 in [GW98]. For $m_0 \in S^{n-1}$, let $2a = \sup_{x\in S^{n-1}} \rho(x)(1 - m_0 \cdot x)$ and assume that this supremum is attained at $x_0 \in S^{n-1}$. Then $P(m_0, a)$ is a supporting paraboloid to $\mathcal{A}$ at x_0, and $\rho^*(m_0) = \dfrac{1}{2a}$. By Remark 2.4, one concludes that $\dfrac{1}{2r_2} \leq \rho^*(m_0) \leq \dfrac{1}{2r_1}$ and $\mathcal{N}_{\mathcal{A}}(S^{n-1}) = S^{n-1}$.

Let $m_0 \in \mathcal{N}_{\mathcal{A}}(x_0)$. We now prove $x_0 \in \mathcal{N}_{\mathcal{A}^*}(m_0)$. Denote by $P(m_0, a)$ the supporting paraboloid to $\mathcal{A}$ at $x_0\rho(x_0)$ with the axis direction m_0. Then $\rho(x)(1-m_0\cdot x)$ attains the maximum $2a$ at x_0 and $\rho^*(m_0) = \dfrac{1}{\rho(x_0)(1 - m_0 \cdot x_0)}$. Obviously, $\rho^*(m) \leq \dfrac{1}{\rho(x_0)(1 - m \cdot x_0)}$ for $m \in S^{n-1}$. Therefore, $P(x_0, \frac{1}{2\rho(x_0)})$ is a supporting paraboloid to $\mathcal{A}^*$ at $m_0\rho^*(m_0)$, and $x_0 \in \mathcal{N}_{\mathcal{A}^*}(m_0)$.

Now, given $m_0 \in S^{n-1}$, there exists $x_0 \in S^{n-1}$ such that $m_0 \in \mathcal{N}_{\mathcal{A}}(x_0)$. Hence, $x_0 \in \mathcal{N}_{\mathcal{A}^*}(m_0)$ and $\mathcal{A}^*$ is an admissible antenna. □

LEMMA 5.3. *Let* $\mathcal{A}_k = \{x\rho_k(x) : x \in S^{n-1}\}$*,* $k \geq 1$*, be a sequence of admissible antennas satisfying* (2.1)*. Assume that* $\mathcal{A}_k$ *converges to the antenna* $\mathcal{A} = \{x\rho(x) : x \in S^{n-1}\}$ *under the Hausdorff metric as* $k \longrightarrow \infty$*. Then* $\mathcal{A}_k^*$ *also converges to* $\mathcal{A}^*$ *under the Hausdorff metric.*

Proof. By Lemma 5.2, $1/(2r_2) \leq \rho_k^* \leq 1/(2r_1)$. So there exists $\eta_0 > 0$ such that $\rho_k^*(m) = \inf_{1-m\cdot x\geq\eta_0} \dfrac{1}{\rho_k(x)(1 - m \cdot x)}$. We obtain that

$$|\rho_k^*(m) - \rho^*(m)| \leq \frac{1}{\eta_0} \sup_{1-m\cdot x\geq\eta_0} \left|\frac{1}{\rho_k(x)} - \frac{1}{\rho(x)}\right|.$$

It follows from Remark 4.3 that ρ_k^* converges to ρ^* uniformly on S^{n-1}. The lemma is proved. □

We now establish the following important lemma about the Legendre transform of antennas in the setting of weak solutions.

LEMMA 5.4. *Let* $\mathcal{A} = \{x\rho(x) : x \in S^{n-1}\}$ *be an admissible antenna such that* (2.1) *and* (3.1) *hold. Then* $\mathcal{A}^*$ *satisfies*

$$\Lambda^{-1}|E^*| \leq |\mathcal{N}_{\mathcal{A}^*}(E^*)| \leq \lambda^{-1}|E^*|, \tag{5.1}$$

for all Borel subsets $E^* \subset S^{n-1}$*.*

Proof. Let $E = \mathcal{N}_{\mathcal{A}}^{-1}(E^*) = \{x \in S^{n-1} : \exists\, m \in E^*$ such that $m \in \mathcal{N}_{\mathcal{A}}(x)\}$. We first prove

Claim 1. $E^* \subset \mathcal{N}_{\mathcal{A}}(E)$ and $|\mathcal{N}_{\mathcal{A}}(E) \setminus E^*| = 0$.

It is easy to see by definition that $E^* \subset \mathcal{N}_{\mathcal{A}}(E)$. Let

$$M = \{x \in S^{n-1} : \mathcal{A} \text{ is not differentiable at } x\rho(x)\}.$$

We have $|M| = 0$. We claim that $\mathcal{N}_{\mathcal{A}}(E) \setminus E^* \subset \mathcal{N}_{\mathcal{A}}(M)$, and then from (3.1) we obtain that $|\mathcal{N}_{\mathcal{A}}(E) \setminus E^*| = 0$. To prove the claim, given $y \in E$, let $m \in \mathcal{N}_{\mathcal{A}}(y) \setminus E^*$. By definition of E, $\mathcal{N}_{\mathcal{A}}(y) \cap E^* \neq \emptyset$, and therefore there is $m_0 \in \mathcal{N}_{\mathcal{A}}(y) \cap E^*$, $m_0 \neq m$. Therefore, $\mathcal{A}$ has at least two supporting paraboloids and hence two supporting hyperplanes at $y\rho(y)$, and so is not differentiable at $y\rho(y)$, which proves the claim.

Claim 2. $E \subset \mathcal{N}_{\mathcal{A}^*}(E^*)$ and $|\mathcal{N}_{\mathcal{A}^*}(E^*) \setminus E| = 0$.

Given $x \in E$, by definition of E there exists $m \in E^*$ such that $m \in \mathcal{N}_{\mathcal{A}}(x)$. By Lemma 5.2, $x \in \mathcal{N}_{\mathcal{A}^*}(m)$, and hence $E \subset \mathcal{N}_{\mathcal{A}^*}(E^*)$.

Let $y \in \mathcal{N}_{\mathcal{A}^*}(E^*) \setminus E$. Then $y \in \mathcal{N}_{\mathcal{A}^*}(m_0)$ for some $m_0 \in E^*$. By Claim 1, there exists $x_0 \in E$ such that $m_0 \in \mathcal{N}_{\mathcal{A}}(x_0)$. Since $y \neq x_0$, by Theorem 4.2, $\mathcal{N}_{\mathcal{A}}(y) \cap \mathcal{N}_{\mathcal{A}}(x_0) = \emptyset$. If $m_1 \in \mathcal{N}_{\mathcal{A}}(y)$, then $m_1 \neq m_0$. By Lemma 5.2, $y \in \mathcal{N}_{\mathcal{A}^*}(m_1)$. Therefore, $y \in \mathcal{N}_{\mathcal{A}^*}(m_0) \cap \mathcal{N}_{\mathcal{A}^*}(m_1)$. From Lemma 5.2, this implies that $m_0, m_1 \in \mathcal{N}_{\mathcal{A}^{**}}(y)$, where $\mathcal{A}^{**}$ is the Legendre transform of $\mathcal{A}^*$. So, $\mathcal{A}^{**}$ is not differentiable at $y\rho^{**}(y)$ where ρ^{**} is the radial function of $\mathcal{A}^{**}$. We conclude that $\mathcal{N}_{\mathcal{A}^*}(E^*) \setminus E$ is a subset of the set where $\mathcal{A}^{**}$ is not differentiable and which has measure zero. This proves Claim 2.

To finish the proof, from Claims 1 and 2 we get $|E^*| = |\mathcal{N}_{\mathcal{A}}(E)|$ and $|\mathcal{N}_{\mathcal{A}^*}(E^*)| = |E|$, and so (5.1) follows from (3.1). □

6. C^1 regularity

We are now ready to prove the C^1 regularity for weak solutions of the antenna problem. First show the following lemma.

LEMMA 6.1. *If* $\mathcal{A} = \{x\rho(x) : x \in S^{n-1}\}$ *is an admissible antenna satisfying* (2.1) *and* (3.1), *then* $\mathcal{N}_{\mathcal{A}}$ *is a homeomorphism from* S^{n-1} *onto* S^{n-1} *with* $\mathcal{N}_{\mathcal{A}}^{-1} = \mathcal{N}_{\mathcal{A}^*}$. *Moreover*, $\{\mathcal{N}_{\mathcal{A}}\}$ *is equicontinuous*, *i.e.*, *there exists an increasing continuous function* σ *with* $\lim_{h\to 0^+} \sigma(h) = 0$ *such that* $|\mathcal{N}_{\mathcal{A}}(x) - \mathcal{N}_{\mathcal{A}}(y)| \leq \sigma(|x-y|)$, *where* σ *depends only on* r_1, r_2, λ, Λ, *and* n.

Proof. We first prove that $\mathcal{N}_{\mathcal{A}}$ is single-valued. Otherwise, if $m_1, m_2 \in \mathcal{N}_{\mathcal{A}}(x_0)$ with $m_1 \neq m_2$, then by Lemma 5.2, $x_0 \in \mathcal{N}_{\mathcal{A}^*}(m_1) \cap \mathcal{N}_{\mathcal{A}^*}(m_2)$. On the other hand, by Lemmas 5.2 and 5.4, $\mathcal{A}^*$ satisfies (2.1) and (3.1) with different constants, and so by applying Theorem 4.2 to $\mathcal{A}^*$ we get $\mathcal{N}_{\mathcal{A}^*}(m_1) \cap \mathcal{N}_{\mathcal{A}^*}(m_2) = \emptyset$, a contradiction. In light of Theorem 4.2, and since $\mathcal{N}_{\mathcal{A}}(S^{n-1}) = S^{n-1}$, see the proof of Lemma 5.2, we obtain that $\mathcal{N}_{\mathcal{A}}$ is bijective. If $m = \mathcal{N}_{\mathcal{A}}(x)$, then $x = \mathcal{N}_{\mathcal{A}^*}(m)$. This proves $\mathcal{N}_{\mathcal{A}}^{-1} = \mathcal{N}_{\mathcal{A}^*}$.

It remains to show that $\{\mathcal{N}_{\mathcal{A}}\}$ is equicontinuous. Assume by contradiction that there exist $\mathcal{A}_k$, x_k, y_k, and $\varepsilon_0 > 0$ such that $|x_k - y_k| \longrightarrow 0$ and $|\mathcal{N}_{\mathcal{A}_k}(x_k) - \mathcal{N}_{\mathcal{A}_k}(y_k)| \geq \varepsilon_0$. By compactness and Corollary 4.5, replaced by a subsequence if necessary, we may assume that $x_k \longrightarrow z_0$, $y_k \longrightarrow z_0$, $m_k = \mathcal{N}_{\mathcal{A}_k}(x_k) \longrightarrow m_0$, $m_k' = \mathcal{N}_{\mathcal{A}_k}(y_k) \longrightarrow m_0'$, and $\mathcal{A}_k \longrightarrow \mathcal{A}_0$. It is easy to verify that $m_0 \in \mathcal{N}_{\mathcal{A}_0}(z_0)$ and $m_0' \in \mathcal{N}_{\mathcal{A}_0}(z_0)$. Since $\mathcal{N}_{\mathcal{A}_0}$ is single-valued, $m_0 = m_0'$ which contradicts the assumption $|m_0 - m_0'| \geq \varepsilon_0$. We thus prove the lemma. □

THEOREM 6.2. *If* $\mathcal{A} = \{x\rho(x) : x \in S^{n-1}\}$ *is an admissible antenna satisfying* (2.1) *and* (3.1), *then* $\mathcal{A}$ *and* $\mathcal{A}^*$ *are* C^1, *with* C^1 *modulus of continuity depending only on* r_1, r_2, λ, Λ, *and* n.

Proof. If $P(m,b)$ is a supporting paraboloid to $\mathcal{A}$ at $x\rho(x)$, then by the Snell law $\mathcal{A}$ has a supporting hyperplane at $x\rho(x)$ with the inward normal $\dfrac{m-x}{|m-x|}$. By Lemma 6.1, this field of inward normals for $\mathcal{A}$ is continuous.

It remains to show that $\mathcal{A}$ is differentiable and hence has only one supporting hyperplane at each point. Let $Y = (y_1, \cdots, y_n) \in \mathcal{A}$ and assume that $\{z_n = y_n\}$ is the equation of a supporting hyperplane Π_0 to $\mathcal{A}$ at Y. For $X \in \mathcal{A}$ near Y and without loss of generality we can write $X - Y = x_1 e_1 + x_n e_n$ with x_1, $x_n > 0$. By the continuity of the inward normals mentioned before, there exists a supporting hyperplane at X with the equation $\nu(X) \cdot (z - X) = 0$, where the inward normal $\nu(X) = (\nu_1(X), \cdots, \nu_n(X))$ is close to $\nu(Y) = e_n$. From the convexity, $\nu(X) \cdot (Y - X) \geq 0$, and so $x_n \leq -\dfrac{\nu_1(X)}{\nu_n(X)} x_1 \leq \varepsilon x_1$, i.e, $\operatorname{dist}(X, \Pi_0) \leq \varepsilon |X - Y|$. Therefore, Π_0 is the tangent plane to $\mathcal{A}$ at Y and the only supporting hyperplane at Y. Since the field of inward normals of tangent planes to $\mathcal{A}$ is continuous, one concludes that $\mathcal{A}$ is of class C^1. The proof is complete. □

COROLLARY 6.3. *If* $\mathcal{A}$ *is a weak solution in the sense of Definition* 2.6 *of the reflector antenna problem with input illumination intensity* $f(x)$ *and output illumination intensity* $g(m)$ *where* $0 < \lambda \leq f(x) \leq \Lambda$ *and* $\lambda \leq g(m) \leq \Lambda$ *on* S^{n-1}, *then* $\mathcal{A}$ *is a* C^1 *antenna.*

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