REYNOLDS AVERAGED NAVIER STOKES (RANS) EQUATIONS

EQUATION DERIVATION

We seek to derive a set of equations for averaged quantities in a turbulent flow.

Notation:

\[
\begin{align*}
  u_x &= \frac{\partial u}{\partial x} \\
  u_{xx} &= \frac{\partial^2 u}{\partial x^2} \\
  u_t &= \frac{\partial u}{\partial t} \\
  p_x &= \frac{\partial p}{\partial x}
\end{align*}
\]

For a Newtonian incompressible fluid, the Navier Stokes equation along \( x \) is:

\[
\rho \left( u_t + uu_x + vu_y + wu_z \right) = -p_x + \mu \left( u_{xx} + u_{yy} + u_{zz} \right)
\]

The terms on the left-hand side can be reformulated as follows:

\[
\Rightarrow
\]

Using the continuity equation for an incompressible flow:

\[
\]

Therefore, the NS equation can be rewritten:

\[
\]

Now, let’s take the time average, term by term. The typical term generated following this procedure is:

\[
\]

Treating all terms in this way, the result is:

\[
\]
Expand the convective term:

Therefore, the final averaged momentum equation along the \( x \) – direction is:

\[ \text{Reynolds-averaged Navier-Stokes (RANS) equation} \]

**CLOSURE PROBLEM**

The averaging of the Navier-Stokes equations yields:

- 3 averaged momentum equations (RANS/\( x \), RANS/\( y \), RANS/\( z \))
- 1 averaged continuity equation
- 10 unknowns

\[ \Rightarrow 4 \text{ equation, 10 unknowns (closure problem)} \]

**REYNOLDS STRESS TENSOR**

Let’s look at the units of the additional unknowns resulting from the averaging procedure:

Based on this analysis, the additional unknowns are grouped under a Reynolds stress tensor:

\[
\tau' = \begin{bmatrix}
-\rho \bar{u'} u' & -\rho \bar{u'} v' & -\rho \bar{u'} w' \\
-\rho \bar{v'} u' & -\rho \bar{v'} v' & -\rho \bar{v'} w' \\
-\rho \bar{w'} u' & -\rho \bar{w'} v' & -\rho \bar{w'} w'
\end{bmatrix}
\]