**UPWIND DIFFERENCING SCHEME EXAMPLE**

1D CONVECTION-DIFFUSION EQUATION

\[
\frac{d}{dx}(\rho u \phi) = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right)
\]

Discretized equation:

\[
(\rho u A)_{e} - (\rho u A)_{w} = \left(\Gamma A \frac{d\phi}{dx}\right)_{e} - \left(\Gamma A \frac{d\phi}{dx}\right)_{w}
\]

With the notation adopted earlier \((F = \rho u \text{ and } D = \frac{\Gamma}{\delta x})\):

\[
F_e \phi_e - F_w \phi_w = D_e (\phi_E - \phi_P) - D_w (\phi_P - \phi_W)
\]

**UPWIND DIFFERENCING SCHEME SUMMARY**

\[a_p \phi_p = a_w \phi_w + a_e \phi_E\]

with:

\[
\begin{align*}
a_p &= a_w + a_e + (F_e - F_w) \\
a_w &= D_w + \max(F_w, 0) \\
a_e &= D_e + \max(0, -F_e)
\end{align*}
\]

**PROBLEM DESCRIPTION**

A property \(\phi\) is transported by means of convection and diffusion through the one-dimensional domain sketched below.

\[\phi = \begin{cases} 1 \quad & x = 0 \\ \phi = 0 \quad & x = L \end{cases}\]

The governing equation is:

\[
\frac{d}{dx}(\rho u \phi) = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right),
\]

and the boundary conditions are prescribed as:

\[
\begin{cases}
\phi_0 = 1 \text{ at } x = 0 \\
\phi_L = 0 \text{ at } x = L
\end{cases}
\]
Using five equally spaced cells and the upwind differencing scheme for convection and diffusion, calculate the distribution of $\phi(x)$ and compare the results with the analytical solution.

**Discretization grid:**

![Discretization grid diagram](image)

**Equation discretization:**
For the case of a positive flow direction, the discretized equation at internal nodes (2, 3, 4) is:

$$a_p \phi_p = a_w \phi_w + a_e \phi_e,$$

where:

$$\begin{align*}
a_p &= a_w + a_e + (F_e - F_w) \\
a_w &= D_w + F_w \\
a_e &= D_e
\end{align*}$$

Assuming a uniform velocity and a uniform diffusion coefficient over the entire domain:

$$\Rightarrow$$

**Boundary conditions:** CVs 1 and 5 require a special treatment since they are adjacent to the domain boundaries.

![Boundary conditions diagram](image)

At node 1, the use of upwind differencing for the convective terms gives:
Similarly at node 5:

Discretized equations at boundary nodes (1, 5):

where:

Solution: To investigate the solution, let’s consider the two cases:

- Case 1: \( u = 0.1 \text{ m/s}, \quad F = \rho u = 0.1, \quad D = \Gamma / \delta x = 0.1/0.2 = 0.5 \Rightarrow \)

![Graph showing numerical solution and exact solution for \( u = 0.1 \text{ m/s} \)]

UDS produces good results at this Peclet number.
• Case 2: \( u = 2.5 \text{ m/s}, \ F = \rho u = 2.5, \ D = \Gamma / \delta x = 0.1/0.2 = 0.5 \Rightarrow \)

As compared to CDS with the same grid resolution (see handout 4.3), UDS produces a more realistic solution, except near the boundary B.