DIFFUSION EQUATION EXAMPLE: 1D STEADY STATE HEAT CONDUCTION (NO SOURCE, TEMPERATURE AND HEAT FLUX PRESCRIBED AT BOUNDARIES)

PROBLEM DESCRIPTION
In this example problem, we will consider the application of the finite volume method to the solution of a simple diffusion problem involving the cooling of a circular fin by means of convective heat transfer along its length. Convection gives rise to a temperature-dependent heat loss term in the governing equation:

\[
\frac{d}{dx}\left(kA \frac{dT}{dx}\right) - hP(T - T_\infty) = 0
\]

where \(k\) is the thermal conductivity, \(T\) (dependent variable) is the temperature, \(A\) is the fin cross-sectional area, \(P\) is the perimeter, and \(T_\infty\) is the ambient temperature.

The base of the fin is at temperature \(T_B = 100^\circ\text{C}\), and the end is insulated. The fin is exposed to an ambient temperature \(T_\infty = 20^\circ\text{C}\). Lastly, \(hP/kA = 25\ \text{m}^2\).

Calculate the temperature distribution along the fin.

**Governing equation (differential form):**

**Governing equation (integral form):**

**Grid generation:** In this solution, we will divide the length of the fin into five equal control volumes.
**Equation discretization:** For nodes 2, 3 and 4, the discretized governing equation for CVs surrounding these nodes is:

\[
\begin{align*}
\text{Discretized equation: } & \quad \text{which can be rewritten as:} \\
\text{Boundary conditions: } & \quad \text{Nodes 1 and 5 are boundary nodes, and therefore require a special treatment.} \\
\text{For the CV surrounding node 1: } & \quad \text{Discretized equation:}
\end{align*}
\]

The discretized equation valid for the first CV can be rewritten in a form similar to that valid in the other CVs:
where:

For the CV surrounding node 5:
At node 5, the flux across the east boundary is zero since the east side of the CV is an insulated boundary.

Discretized equation:

which can be rewritten as:

Discretized equation for boundary node 5:

where:

Solution: This set of equations can be rewritten in matrix form:

\[
\begin{bmatrix}
20 & -5 & 0 & 0 & 0 \\
-5 & 15 & -5 & 0 & 0 \\
0 & -5 & 15 & -5 & 0 \\
0 & 0 & -5 & 15 & -5 \\
0 & 0 & 0 & -5 & 10 \\
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
\end{bmatrix} =
\begin{bmatrix}
1100 \\
100 \\
100 \\
100 \\
100 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
\end{bmatrix} =
\begin{bmatrix}
64.22 \\
36.91 \\
26.50 \\
22.60 \\
21.30 \\
\end{bmatrix}
\]