Problem 1

1- The general form of the heat diffusion equation is:

\[ \rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + \dot{e}_g \]

Assumptions:

- one-dimensional conduction (along the \( x \)-direction only): \( \nabla = \frac{\partial}{\partial x} \mathbf{i} \) and \( T = T(x) \)
- steady state conditions: \( \frac{\partial}{\partial t} = 0 \)
- constant thermal properties: \( k = \text{constant} \)
- no heat generation within the base plate material: \( \dot{e}_g = 0 \)

Therefore, the reduced heat conduction equation is:

\[ k \frac{\partial^2 T}{\partial x^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 T}{\partial x^2} = 0 \]

Integrating with respect to \( x \):

\[ \frac{\partial T}{\partial x} = C_1, \]

where \( C_1 \) is a constant of integration. We integrate a second time to obtain an expression for the temperature distribution:

\[ T(x) = C_1 x + C_2, \]

where \( C_2 \) is another constant of integration.

To solve for the two constants, we need to consider the boundary conditions in this problem.
- **BC1**: the heat flux along the lower surface is equal to \( \dot{q}_m \)

\[ \Rightarrow -k \frac{\partial T}{\partial x} \bigg|_{x=0} = \dot{q}_m \quad \Rightarrow -k C_1 = \dot{q}_m \quad \Rightarrow \quad C_1 = -\dot{q}_m / k \]

- **BC2**: the conductive heat flux along the top surface is equal to the convective heat flux

\[ \Rightarrow -k \frac{\partial T}{\partial x} \bigg|_{x=L} = h(T(L) - T_\infty) \quad \Rightarrow -k C_1 = h \left[ (C_1 L + C_2) - T_\infty \right] \quad \Rightarrow \quad C_2 = T_\infty - \left( \frac{k + h L}{h} \right) C_1 \]
Substituting back in the general form of the temperature distribution:

\[
T(x) = \frac{\dot{q}_{in}}{k} \left( \frac{k}{h} + L - x \right) + T_\infty
\]

**Numerical applications:**

\[
T_0 = T(0) = \frac{6000}{13.5} \left( \frac{13.5}{20} + 0.01 \right) + 35 \quad \Rightarrow \quad T_0 = 339^\circ C
\]

\[
T_L = T(L) = \frac{6000}{13.5} \left( \frac{13.5}{20} \right) + 35 \quad \Rightarrow \quad T_L = 335^\circ C
\]

2. \( T_L = 335^\circ C > 200^\circ C \): the outer surface temperature is 135°C higher than the safe temperature of 200°C. The outer surface of the engine should be covered with protective insulation to prevent fire hazard in the event of oil leakage.

**Problem 2**

In order to determine the inside surface temperature of the furnace \( T_0 \), the temperature distribution within the furnace must be determined. This temperature distribution depends on the outer surface temperature \( T(L) \), which can be determined by considering an energy balance on the outer surface.

Along that surface, heat is entering through a uniform flux \( \dot{q}_{in} \). Heat is lost in the same amount via convection and radiation. The energy balance can then be written:

\[
\dot{q}_{in} = h(T(L) - T_\infty) + \varepsilon \sigma (T(L)^4 - T_{sur}^4)
\]

**Numerical application:**

\[
5000 = 10 \times [T_L - (273 + 20)] + 0.3 \times (5.67 \times 10^{-8}) \times [T_L^4 - (20 + 273)^4]
\]

Using a numerical solver on the calculator:

\[
\Rightarrow T_L = 594 \text{ K}
\]

With the knowledge of this boundary condition, we can now derive an expression for the temperature distribution in the furnace front.

The general form of the heat diffusion equation is:

\[
\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + \dot{e}_s
\]

Assumptions:

- one-dimensional conduction (along the \( x \)-direction only): \( \nabla = \frac{\partial}{\partial x} \hat{i} \) and \( T = T(x) \)
- steady state conditions: \( \frac{\partial}{\partial t} = 0 \)
- constant thermal properties: \( k = \text{constant} \)
- no heat generation: \( \dot{q} = 0 \)

Therefore, the reduced heat conduction equation is:

\[
k \frac{\partial^2 T}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 T}{\partial x^2} = 0
\]

Integrating with respect to \( x \):

\[
\frac{\partial T}{\partial x} = C_1,
\]

where \( C_1 \) is a constant of integration. We integrate a second time to obtain an expression for the temperature distribution:

\[
T(x) = C_1 x + C_2,
\]

where \( C_2 \) is another constant of integration.

To solve for the two constants, we need to consider the boundary conditions in this problem.
- BC1: the heat flux along the left surface is equal to \( \dot{q}_m \)

\[
\Rightarrow -k \frac{\partial T}{\partial x} \bigg|_{x=0} = \dot{q}_m \Rightarrow -k C_1 = \dot{q}_m \Rightarrow C_1 = -\dot{q}_m / k
\]

- BC2: the temperature along the right surface is \( T_L \)

\[
\Rightarrow C_1 L + C_2 = T_L \Rightarrow C_2 = T_L - C_1 L
\]

Substituting back in the general form of the temperature distribution:

\[
T(x) = \frac{\dot{q}_m}{k} (L - x) + T_L
\]

**Numerical application:**

\[
T_0 = T(0) = \frac{5000}{25} \times 0.02 + 594 \Rightarrow T_0 = 598 \text{ K}
\]
\[ T_L = T(L) = \frac{6000}{13.5} \left( \frac{13.5}{20} \right) + 35 \Rightarrow T_L = 335^\circ C \]

**Problem 3**

The general form of the heat diffusion equation in spherical coordinates is:

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} \left( k \frac{\partial T}{\partial \phi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( k \sin \theta \frac{\partial T}{\partial \theta} \right) + \dot{q}_{gen} = \rho c \frac{\partial T}{\partial t}
\]

Assumptions:

- one-dimensional conduction (along the \( r \)-direction only): \( \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \phi} = 0 \) and \( T = T(r) \)
- steady state conditions: \( \frac{\partial}{\partial t} = 0 \)
- constant thermal properties: \( k = \text{constant} \)
- no heat generation: \( \dot{q}_{gen} = 0 \)

Therefore, the reduced heat conduction equation is:

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) = 0 \Rightarrow \frac{k}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) = 0 \Rightarrow \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) = 0
\]

We integrate both sides with respect to \( r \):

\[
r^2 \frac{\partial T}{\partial r} = C_1 \Rightarrow \frac{\partial T}{\partial r} = \frac{C_1}{r^2}
\]

We integrate again with respect to \( r \):

\[
T(r) = -\frac{C_1}{r} + C_2
\]

- BC1: at \( r = r_2 \), \( \dot{q} = \dot{q}_{\text{elec}} \)

The heat transferred to the spherical container is due to the electric heater, which generates a heat flux along the \( -\hat{r} \) direction across a surface area \( A = 4\pi r_2^2 \):

\[
\dot{q}_{\text{elec}} = -\frac{\dot{Q}_{\text{elec}}}{4\pi r_2^2}
\]

Since 10% of the electrical power is lost to the insulation, 90% is effectively transferred to the container:
\[ \dot{q}_{\text{elec}} = -\frac{0.9P}{4\pi r_2^2} \]

Therefore, the boundary condition on the outer surface of the container is expressed as:

\[ -k \frac{\partial T}{\partial r}
_{r=r_2} = -\frac{0.9P}{4\pi r_2^2} \Rightarrow -k \frac{C_1}{r_2^2} = -\frac{0.9P}{4\pi r_2^2} \Rightarrow C_1 = \frac{0.9P}{4\pi k} \]

- BC2: \( T(\eta) = T_1 \)

\[ \Rightarrow -\frac{C_1}{r_1} + C_2 = T_1 \Rightarrow C_2 = T_1 + \frac{C_1}{r_1} \Rightarrow C_2 = T_1 + \frac{0.9P}{4\pi kr_1} \]

Substituting back into the temperature distribution:

\[ T(r) = T_1 + \frac{0.9P}{4\pi k} \left( \frac{1}{r_1} - \frac{1}{r} \right) \]

Numerical application:

\[ T_2 = T(r_2) = 120 + \frac{0.9 \times 800}{4\pi \times 1.5} \left( \frac{1}{0.4} - \frac{1}{0.41} \right) \Rightarrow T_2 = 122.3^\circ C \]

The rate of heat supply to the water is calculated as:

\[ \dot{Q}_{\text{elec}} = \dot{m} c_p \Delta T \Rightarrow \dot{m} = \frac{\dot{Q}_{\text{elec}}}{c_p \Delta T} \Rightarrow \dot{m} = \frac{0.9P}{c_p (T_1 - T_w)} \]

Numerical application:

\[ \dot{m} = \frac{0.9 \times 800}{4185 \times (120 - 100)} \Rightarrow \dot{m} = 0.002151 \text{ kg/s} = 7.74 \text{ kg/h} \]