PROBLEM 1: POTENTIAL FLOWS (10 pts)

Consider the two-dimensional flow field:

$$\mathbf{V} = (ax + b) \mathbf{i} + (-ay + c) \mathbf{j}$$

where \(a, b, c\) are constants.

a) (2 pts) Is this flow incompressible?

Continuity equation for incompressible flow:

$$\nabla \cdot \mathbf{V} = 0$$

$$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x} (ax + b) + \frac{\partial}{\partial y} (-ay + c) = a + (-a) = 0$$

$$\Rightarrow \text{The flow is incompressible.}$$

b) (2 pts) Derive an expression for the \(x\) component of the acceleration \(a_x(x, y)\)

$$a_x = \frac{D}{Dt} = \frac{\partial u}{\partial t} + (\nabla \cdot \mathbf{V}) u = \frac{\partial u}{\partial t} + (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) u$$

$$= 0 + u \frac{\partial}{\partial x} (ax + b) + v \frac{\partial}{\partial y} (ax + b)$$

$$\Rightarrow a_x = (ax + b) a$$

c) (4 pts) Generate an expression for the streamfunction \(\psi(x, y)\).

Hint: \(\partial \psi / \partial y = u; \partial \psi / \partial x = -v\)

$$\frac{\partial \psi}{\partial y} = ax + b \Rightarrow \psi(x, y) = axy + by + f(x)$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} (axy + by + f(x)) = ay + f'(x) = ay - c$$

$$\Rightarrow f'(x) = -c$$

$$\Rightarrow f(x) = -cx + D \quad \text{arbitrary constant, set equal to 0}$$

$$\Rightarrow \psi(x, y) = axy + by - cx$$
d) \(2\) pts\) Find the equation for the flow streamline passing through the point \((1, 1)\). Put your answer in the form \(y = y(a, b, c, x)\).

\[
\begin{align*}
\psi(1, 1) &= a + b - c \\
\psi(x, y) &= \psi(1, 1) \\
&\implies ax + by - cx = a + b - c \\
&\implies y(ax + b) = cx + a + b - c \\
&\implies y = \frac{cx + a + b - c}{ax + b}
\end{align*}
\]
PROBLEM 2: LOCAL FLOW ANALYSIS (10 pts)

Consider blood flow through an inclined circular pipe of diameter \( d \). The flow which is driven by a pressure gradient \( \frac{\partial p}{\partial z} = \text{constant} \) is characterized by a volume flow rate \( Q \). Blood properties are defined by the density \( \rho \) and the dynamic viscosity \( \mu \). By performing a local analysis, we wish to derive expressions for the velocity profile, the volume flow rate and the wall shear stress.

![Diagram of blood flow through an inclined circular pipe](image)

The Navier-Stokes and continuity equations in cylindrical coordinates are given below.

\[
\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} \right) = \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right] - \frac{\partial p}{\partial r} + \rho g_r \tag{i}
\]

\[
\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} \right) = \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] - \frac{1}{r} \frac{\partial p}{\partial \theta} + \rho g_\theta \tag{ii}
\]

\[
\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{r} \frac{\partial v_z}{\partial r} \right) + \frac{\partial^2 v_z}{\partial z^2} \right] - \frac{\partial p}{\partial z} + \rho g_z \tag{iii}
\]

\[
\begin{align*}
1 \frac{\partial}{\partial r} \left( r v_r \right) + \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} &= 0 \\
\frac{\partial v_r}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r v_r \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} &= 0 \\
\frac{\partial v_\theta}{\partial \theta} &= 0 \\
\frac{\partial v_z}{\partial z} &= 0
\end{align*}
\tag{iv}
\]

a) (2 pts) Simplify these equations using the following assumptions: (1) steady flow, (2) purely axial flow, (3) axisymmetric flow, and (4) fully developed. On the equations above, indicate the assumption used to eliminate each term.
b) (4 pts) Using the reduced form of equation (iii) and appropriate boundary conditions, derive an expression for the axial velocity component \(v_z\) and express your result in the form:

\[ v_z = v_z(\mu, \partial p/\partial z, r, g, \alpha). \]

\[
(iii) : \quad 0 = \mu \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) - \frac{\partial p}{\partial z} + \rho g z
\]

\[
\Rightarrow \quad \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) = \frac{\mu}{r} \left( \frac{\partial p}{\partial z} - \rho g \sin \alpha \right)
\]

Integrate:

\[
\frac{r}{2} \frac{\partial v_z}{\partial r} = \frac{\mu}{2} \left( \frac{\partial p}{\partial z} - \rho g \sin \alpha \right) + C_1
\]

\[
\Rightarrow \quad v_z(r) = \frac{\mu}{4} \left( \frac{\partial p}{\partial z} - \rho g \sin \alpha \right) + C_1 \ln r + C_2
\]

**BC1** \(v_z\) exists @ \(r = 0\) \(\Rightarrow C_1 = 0\).

**BC2** \(v_z(r = d/2) = 0\) (\(v_z = 0\) at the wall)

\[
0 = \frac{d^2}{16 \mu} \left( \frac{\partial p}{\partial z} - \rho g \sin \alpha \right) + C_2
\]

\[
\Rightarrow C_2 = -\frac{d^2}{16} \left( \frac{\partial p}{\partial z} - \rho g \sin \alpha \right)
\]

\[
\Rightarrow \quad v_z(r) = \frac{1}{4 \mu} \left( \frac{r^2 - \frac{d^2}{4}}{4} \right) \left( \frac{\partial p}{\partial z} - \rho g \sin \alpha \right)
\]
c) **(2 pts)** With the knowledge of the velocity profile, derive an expression for the volume flow rate $Q$ through the blood vessel.

\[
Q = \int_A \nu_z \, dA
\]

\[
\Rightarrow Q = \int_{n=0}^{d/2} 2\pi n \nu_z \, dn
\]

\[
\Rightarrow Q = \frac{\pi}{2\mu} \left( \frac{\partial p}{\partial z} - \rho g \sin x \right) \int_{0}^{d/2} \left( r^3 - r \frac{d^2}{4} \right) \, dn
\]

\[
\Rightarrow Q = \frac{\pi}{2\mu} \left( \frac{\partial p}{\partial z} - \rho g \sin x \right) \left[ \frac{r^4}{4} - \frac{r^2 d^2}{8} \right]_{0}
\]

\[
\Rightarrow Q = \frac{\pi}{2\mu} \left( \frac{\partial p}{\partial z} - \rho g \sin x \right) \left( \frac{d^4}{64} - \frac{d^4}{32} \right)
\]

\[
\Rightarrow Q = \frac{\pi}{128 \mu} \left( \rho g \sin x - \frac{\partial p}{\partial z} \right)
\]

d) **(2 pts)** Derive an expression for the wall shear stress $\tau_w$ on the surface of the pipe.

\[
\tau_w = \mu \frac{d\nu_z}{dn} \bigg|_{n = d/2}
\]

\[
= \left[ \frac{\pi}{2} \left( \frac{\partial p}{\partial z} - \rho g \sin x \right) \right]_{n = d/2}
\]

\[
\Rightarrow \tau_w = \frac{d}{2} \left( \frac{\partial p}{\partial z} - \rho g \sin x \right)
\]