PROBLEM 1 (10 pts)
A rectangular gate (height $h$; width normal to the page $b$) hinged along its top edge is submerged in a stratified lake. The density of the water varies as a function of depth as: $\rho(z) = \rho_0 e^z$, where $\rho_0$ is the water density at the free surface.

![Diagram of the gate and water density](image)

a) (3 pts) Using the coordinate system shown on the figure, develop an expression for the gage pressure distribution in the lake.

Equation of fluid statics: \[
\frac{dp}{dz} = \rho g
\]
+ sign since $z$-axis is downward.

\[\Rightarrow \frac{dp}{dz} = \rho_0 e^z g\]

Integrate on both sides: \[p(z) = \rho_0 g e^z + C\]

Boundary condition: at $z = 0$, $p = 0$ (gage pressure at free-surface)

\[0 = \rho_0 g + C\]
\[\Rightarrow C = -\rho_0 g\]

Therefore: \[p(z) = \rho_0 g (e^z - 1)\]
b) (5 pts) Develop an expression for the resultant pressure force acting on the rectangular gate.

At depth $z$, the infinitesimal pressure force $dF$ acting on the gate is: $dF = \rho(z)\, dA$, where $dA$ is the area of a rectangular strip of height $dz$ and width $b$:

$$dF = \rho_0 g (e^z - 1) (bdz)$$

Integrate over the gate:

$$F = \int_{z=H}^{z=H+h} \rho_0 g b (e^z - 1) \, dz$$

$$\Rightarrow F = \rho_0 g b (e^{H+h} - e^H - e^z + h)$$

$$\Rightarrow F = \rho_0 g b (e^{H+h} - e^H + h)$$

c) (2 pts) By using an integration approach, generate an expression for the moment $M$ exerted by the water on the gate about the hinge. You may express your final answer in terms of an integral but you don’t have to perform the integration.

Similarly, at depth $z$, the force $dF$ generates a moment $dM$ about the hinge:

$$dM = dF \cdot (z-H)$$

$$dM = \rho_0 g b (e^z - 1) (z-H)$$

$$\Rightarrow M = \int_{H}^{H+h} \rho_0 g b (e^z - 1) (z-H) \, dz$$
**Problem 2 (10 pts)**

The underbody of a Formula One car is designed to enhance the car aerodynamic performance. It can be represented as a two-dimensional rectangular channel (i.e., width normal to the page = 1) of varying height ($h$ at the entrance; $H$ at the exit). At the entrance, the air velocity is approximated as uniform (velocity $U$) while at the exit, it is discharged with a linear velocity distribution: $u(y) = \frac{U_{\text{max}}}{H} y$.

![Diagram of car underbody with airflow](image)

\[ u(y) = \frac{U_{\text{max}}}{H} y \]

(a) **(5 pts)** By performing a control volume analysis, derive an expression for the maximum velocity at the outlet section $U_{\text{max}}$ in terms of $h, H, U$. Show all your steps and state all your assumptions.

**CV definition:** air between the ground and the car underbody.

**Assumptions:**
- Steady flow
- Incompressible flow

**Conservation of mass:**
\[
\frac{\partial}{\partial t} \int_{CV} \rho \, dV + \oint_{CS} \rho \mathbf{V} \cdot d\mathbf{A} = 0
\]

Reduced equation:
\[
- \int_{\text{inlet}} \rho \, u \, dA + \int_{\text{outlet}} \rho \, u(y) \, dA = 0
\]

\[
\Rightarrow - \rho \, u \, h + \rho \, \frac{U_{\text{max}}}{H} \int_{y=0}^{H} y \, dy = 0
\]

\[
\Rightarrow \frac{U_{\text{max}}}{H} \left[ \frac{y^2}{2} \right]_0^H = uh
\]

Conservation of mass:
\[
\frac{\partial}{\partial t} \int_{CV} \rho \, dV + \oint_{CS} \rho \mathbf{V} \cdot d\mathbf{A} = 0
\]

Linear momentum balance:
\[
\frac{\partial}{\partial t} \int_{CV} \rho \mathbf{V} \, dV + \oint_{CS} \rho \mathbf{V} \cdot d\mathbf{A} = \sum \mathbf{F}_{\text{on CV}}
\]

**Note:**
- $CV$: Control Volume
- $CS$: Control Surface
- $\rho$: Density
- $\mathbf{V}$: Velocity vector
- $dV$: Differential volume
- $dA$: Differential area
- $F_{\text{on CV}}$: Forces on the control volume
b) (5 pts) By performing a control volume analysis, derive an expression for the resultant force exerted by the air on the car along the horizontal $x$-direction in terms of $\rho, U, h, H$. You may ignore pressure forces at the entrance and exit of the channel, as well as the viscous force between the air and the ground surface. Show all your steps and state all your assumptions.

- **Using the same CV and assumptions**, and neglecting pressure at inlet/outlet and shear stress at ground.

- **Free-body diagram:**

  ![Free-body diagram](image)

  $(R_x: \text{force exerted by surroundings on CV})$

- **Linear momentum balance:**

  \[
  \frac{d}{dt} \int_{CV} \rho \mathbf{V} \, dV + \oint_{CS} \rho \mathbf{V} \cdot d\mathbf{A} = \sum \mathbf{F}_{on \ CV}
  \]

- Using the assumptions and projecting along $x$-direction,

  \[
  \int_{inlet} \rho \mathbf{V} \cdot d\mathbf{A} + \int_{outlet} \rho \mathbf{V} \cdot d\mathbf{A} = \sum F_x \text{ on CV}
  \]

  \[
  \rho U^2 h + \rho \left( \frac{U_{max}}{H} \right)^2 \int_0^H y^2 \, dy = R_x
  \]

  \[
  \Rightarrow \rho U^2 h + \rho \frac{U_{max}^2}{H^2} \left[ \frac{y^3}{3} \right]_0^H = R_x
  \]

  \[
  \Rightarrow \rho U^2 h + \frac{4}{3} \rho U^2 h^2 = R_x
  \]

  **The force exerted on the car is the opposite of $R_x$:**

  \[
  -R_x = \rho U^2 h - \frac{4}{3} \rho U^2 h^2 = \rho U^2 h \left( 1 - \frac{4h}{3H} \right)
  \]

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Conservation of mass:

\[
\frac{\partial}{\partial t} \int_{CV} \rho dV + \oint_{CS} \rho \mathbf{V} \cdot d\mathbf{A} = 0
\]

Linear momentum balance:

\[
\frac{\partial}{\partial t} \int_{CV} \rho \mathbf{V} \, dV + \oint_{CS} \rho \mathbf{V} \cdot d\mathbf{A} = \sum \mathbf{F}_{on \ CV}
\]