Problem 1

A belt moves upward at velocity $V$, dragging a film of viscous liquid of constant thickness $h$. There is no imposed pressure gradient along the $x$-direction. Near the belt, the film moves upward due to no slip. At its outer edge, the film moves downward due to gravity.

The two-dimensional Navier Stokes equations and the continuity equation are given below in Cartesian coordinates:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + g_x$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + g_y$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

1. Using the following assumptions: (1) steady flow; (2) fully developed flow; and (3) no pressure gradient along $x$; simplify the above equations.

2. Using one of the simplified equations, demonstrate mathematically that there is no flow in the $y$-direction. Clearly state all assumptions and boundary conditions.

3. Using your result from part 2), simplify further the Navier-Stokes equations.

4. Assuming zero shear stress at the outer film edge (i.e., liquid-gas interface), derive an expression for the $x$-component of the velocity. Clearly state all assumptions and boundary conditions.

5. For what value of $V$ does the net volume flow rate per unit depth become equal to zero across the fluid film?
Problem 2
A fountain consists of a pipe (radius $R$) inside which a fluid (density $\rho$, viscosity $\mu$) flows upwards at a constant volume flow rate $Q$. As it comes out of the pipe, the fluid forms a little pool on the top of the cylinder and then flows down covering the exterior of the pipe. After a certain development length, the fluid forms a thin layer of constant thickness $h$ coating the external surface of the pipe. We would like to study the flow in this fully developed flow region (length $L$).

The $z-$component of the Navier Stokes equations and the continuity equation in cylindrical coordinates are given below.

$$\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] - \frac{\partial p}{\partial z} + \rho g_z \tag{i}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left( rv_r \right) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \tag{ii}$$

1- Simplify these equations using the following assumptions: steady flow (1), axisymmetric flow (2), fully developed flow (3), no imposed pressure gradient along the $z-$direction (4). Write the reduced equations below.

2- Using the reduced continuity equation and an appropriate boundary condition, show that there is no radial velocity in this flow. Use this result (5) to simplify further the Navier-Stokes equation.
3- Using appropriate boundary conditions, derive an expression for $v_z$, the $z-$component of the fluid film velocity. You may assume that the interface between the fluid film and the surrounding air is a free surface with no shear stress.

4- Derive an expression for the viscous drag $D$ exerted by the fluid film on the fully developed flow region of the cylinder.

5- Write the equation that you would have to solve in order to obtain a relationship between the volume flow rate $Q$ in the hollow cylinder and the film thickness $h$. Do not solve this equation.