Problem 1

1- Gage pressure distribution

We start from the equation of fluid statics: \( \frac{dp}{dz} = \rho g \) (+ sign since the z-axis is oriented downward).

Substituting the density expression:

\[
\frac{dp}{dz} = \rho_0 g \left(1 + \frac{z}{2}\right)
\]

Integrating both sides with respect to \( z \):

\[
p(z) = \rho_0 g \left(z + \frac{z^2}{4}\right) + C
\]

Since we’re asked to derive the expression for the gage pressure distribution, we express the boundary condition in terms of the gage pressure:

\[p(z = 0) = 0 \Rightarrow C = 0\]

Therefore, the gage pressure distribution is:

\[
p(z) = \rho_0 g z \left(1 + \frac{z}{4}\right)
\]

2- Magnitude of pressure force

At a depth \( z \), the water exerts a small pressure force \( dF \) on the gate:

\[dF = p(z) dA,\]

where \( dA \) is the surface area of a small rectangular region of height \( dz \) and width \( W \) (i.e., \( dA = Wdz \))

Therefore:

\[dF = \rho_0 g z \left(1 + \frac{z}{4}\right) Wdz\]

The resultant pressure force on the gate is obtained by integration:

\[
F = \int_{z=h}^{h+H} \rho_0 g z \left(1 + \frac{z}{4}\right) Wdz = \rho_0 g W \int_{h}^{h+H} \left(z + \frac{z^2}{4}\right) dz = \rho_0 g W \left[ \frac{z^2}{2} + \frac{z^3}{12} \right]_{h}^{h+H}
\]
3- Let’s pick the center of the coordinate system (i.e., point located along the free-surface) as the reference point for the calculation of all moments.

The moment of the pressure force about this point is: \( M = Fz_c \), where \( z_c \) is the location along \( z \) of the center of pressure.

Alternatively, this moment can be calculated by integration. At depth \( z \), the pressure force \( dF \) generate a moment \( dM \) about the reference point:

\[
dM = zdF
\]

The resultant moment is obtained by integration over the gate:

\[
M = \int_{h}^{h+H} \rho_0 g z^2 \left( 1 + \frac{z}{4} \right) Wdz = \rho_0 g W \int_{h}^{h+H} \left( z^3 + \frac{z^4}{4} \right) dz = \rho_0 g W \left[ \frac{z^4}{3} + \frac{z^5}{16} \right]_{h}^{h+H} \]

\[
\Rightarrow M = \rho_0 g W \left[ \frac{(h + H)^3}{3} + \frac{(h + H)^4}{16} - \frac{h^3}{3} - \frac{h^4}{16} \right]
\]

Therefore:

\[
Fz_c = \rho_0 g W \left[ \frac{(h + H)^3}{3} + \frac{(h + H)^4}{16} - \frac{h^3}{3} - \frac{h^4}{16} \right]
\]

\[
\Rightarrow z_c = \frac{(h + H)^3}{3} + \frac{(h + H)^4}{16} - \frac{h^3}{3} - \frac{h^4}{16} - \frac{h^3}{2} - \frac{h^4}{12}
\]

\[
\Rightarrow z_c = \frac{(h + H)^3}{3} + \frac{(h + H)^4}{16} - \frac{h^3}{3} - \frac{h^4}{16} - \frac{h^3}{2} - \frac{h^4}{12}
\]

**Problems 2**

1- Gage pressure distribution

We start from the equation of fluid statics: \( \frac{dp}{dz} = -\rho g \) (\( - \) sign since the \( z \)-axis is oriented upward). Integrating both sides with respect to \( z \):

\[
p(z) = -\rho g z + C
\]

Since we’re asked to derive the expression for the gage pressure distribution, we express the boundary condition in terms of the gage pressure:

\[ p(z = H) = 0 \Rightarrow -\rho g H + C = 0 \Rightarrow C = \rho g H \]
Therefore, the gage pressure distribution is:

\[ p(z) = \rho g (H - z) \]

2- Since the gate is curved, we need to calculate both the horizontal and vertical components of the pressure force.

Calculation of the horizontal component \( F_h \):

To calculate \( F_h \), we calculate the pressure force acting on the vertical projection of the dam. At a depth \( z \), the water exerts a small pressure force \( dF_h \) on the vertical projection of the dam:

\[ dF_h = p(z) dA, \]

where \( dA \) is the surface area of a small rectangular region of height \( dz \) and width equal to 1 normal to the page (\( dA = dz \)). Therefore:

\[ dF_h = \rho g (H - z) dz \]

The resultant horizontal pressure force is obtained by integration:

\[ F_h = \int_{z=0}^{H} \rho g (H - z) dz = \rho g \left[ Hz - \frac{z^2}{2} \right]_0^H \]

\[ \Rightarrow F_h = \frac{\rho g H^2}{2} \]

Calculation of the vertical component \( F_v \):

The vertical component of the pressure force acting on the curved surface (\( F_v \)) is equal to the sum of the pressure force acting on the horizontal projection of the gate (\( F \)) and the weight of the fluid block (\( W \)):

\[ F_v = F + W \]

The horizontal projection of the curved surface is a plane surface of length \( L \) and width equal to 1 normal to the page. Based on the equation describing the dam curvature:

\[ H = 0.2L^2 \Rightarrow L = \sqrt{\frac{H}{0.2}} \]
The gage pressure along the horizontal projection is \( p(z = H) = 0 \). Therefore the pressure force acting on the horizontal projection is also zero:

\[ F = 0 \]

The weight of the fluid block is the weight of the fluid volume located between the horizontal projection, the vertical projection and the curved surface. The surface area of the fluid block is equal to the difference between the surface area of the rectangle of height \( H \) and length \( L \) and the surface area located under the curve:

\[
A_{\text{fluid block}} = HL - \int_{x=0}^{L} 0.2x^2 \, dx = \left( \frac{H^3}{0.2} \right)^{1/2} - 0.2 \left[ \frac{x^3}{3} \right]_0^H
\]

\[ \Rightarrow A_{\text{fluid block}} = \frac{2}{3} \left( \frac{H^3}{0.2} \right)^{1/2} \]

It follows that the weight of the fluid block is:

\[ W = \frac{2}{3} \rho g \left( \frac{H^3}{0.2} \right)^{1/2} \]

Therefore, the vertical component of the pressure force acting on the curved surface is:

\[ F_v = F + W = \frac{2}{3} \rho g \left( \frac{H^3}{0.2} \right)^{1/2} \]

The total magnitude \( F_{\text{tot}} \) of the pressure force is calculated as:

\[ F_{\text{tot}} = \sqrt{F_h^2 + F_v^2} \]

\[ \Rightarrow F_{\text{tot}} = \sqrt{\left( \frac{\rho g H^2}{2} \right)^2 + \left( \frac{2 \rho g}{3} \left( \frac{H^3}{0.2} \right)^{1/2} \right)^2} = \rho g H \sqrt{\frac{H^2}{4} + \frac{4H}{1.8}} \]

3- The angle of the resultant pressure force relative to the horizontal direction is given by:

\[ \tan \alpha = \frac{F_v}{F_h} = \frac{4}{3 \sqrt{0.2H}} \]
Problem 3

The fluid in the container experiences rigid-body motion. Therefore, the governing equation of fluid statics is valid. Assuming a vertical axis (z-axis) oriented upward:

$$-\nabla p - \rho g \hat{k} = \rho a$$

Projecting this equation along the vertical direction: $$\frac{\partial p}{\partial z} - \rho g = \rho a_z$$

Re-arranging:

$$\frac{\partial p}{\partial z} = -\rho (a_z + g)$$

Integrating both sides with respect to $z$:

$$p(z) = -\rho z (a_z + g) + C$$

The constant of integration $C$ can be obtained by considering an appropriate boundary condition. Along the free-surface of the fluid, the gage pressure is equal to 0 (surface exposed to atmospheric pressure):

$$p(z = z_{\text{free surface}}) = 0 \Rightarrow 0 = -\rho z_{\text{free surface}} (a_z + g) + C \Rightarrow C = \rho z_{\text{free surface}} (a_z + g)$$

The location of the free surface corresponds to the total height of fluid in the container, which can be calculated:

$$z_{\text{free surface}} = \frac{V}{A}$$

Therefore: $C = \rho \frac{V}{A} (a_z + g)$, and the gage pressure distribution is expressed as:

$$p(z) = -\rho z (a_z + g) + \rho \frac{V}{A} (a_z + g)$$

$$\Rightarrow p(z) = \rho (a_z + g) \left( \frac{V}{A} - z \right)$$

It follows that the gage pressure at the bottom of the container is:

$$p(0) = \rho (a_z + g) \frac{V}{A}$$