**Problem 1**

**Dimensional analysis**

A functional relationship is to be found between $\tau, \rho, \mu, V, D$ ($n = 5$)

- **Reference dimensions:**
  
  $[\tau] = \frac{M}{LT^2}$ ; $[D] = L$ ; $[V] = \frac{L}{T}$ ; $[\mu] = \frac{M}{LT}$ ; $[\rho] = \frac{M}{L^3}$ ($r = 3$ reference dimensions used)

- The number of pi terms for this analysis is $n - r = 5 - 3 = 2$ pi terms

- Selection of repeating variables: we can choose any variable as a repeating variable, except the dependent variable ($\tau$). Arbitrarily, let’s pick $V, D, \rho$ as repeating variables.

- Pi terms determination

  The first pi term can be written: $\Pi_1 = \tau V^\alpha D^\beta \rho^\gamma$

  In terms of reference dimensions: $[\Pi_1] = \frac{M}{LT^2} \left( \frac{L}{T} \right)^\alpha \left( \frac{M}{L^3} \right)^\beta = M^0 L^0 T^0$

  Equation for M: $1 + \gamma = 0$
  Equation for L: $-1 + \alpha + \beta - 3\gamma = 0$
  Equation for T: $-2 - \alpha = 0$

  Solving for this system: $
  \begin{align*}
  \alpha &= -2 \\
  \gamma &= -1 \\
  \beta &= 0
  \end{align*}$

  Therefore: $\Pi_1 = \frac{\tau}{\rho V^2}$

  The second pi term can be written: $\Pi_2 = \mu V^\alpha D^\beta \rho^\gamma$

  By inspection, we recognize the variables used to form the inverse of the Reynolds number.

  Therefore, directly: $\Pi_2 = \frac{\mu}{\rho V D}$

  We conclude that there exists a relationship between the two pi terms by writing:
\[ \Pi_1 = f(\Pi_2) \Rightarrow \frac{\tau}{\rho V^2} = f\left(\frac{\mu}{\rho V D}\right) = g\left(\frac{\rho V D}{\mu}\right) \]

The function \( g \) can be determined by plotting \( \frac{\tau}{\rho V^2} \) (y-axis) as a function of \( \frac{\rho V D}{\mu} \) (x-axis):

A discontinuity is observed between \( \text{Re} = 2000 \) and \( \text{Re} = 4000 \), which corresponds to the transition regime for pipe flow.

**Problem 2**

1. Dimensional analysis, method of repeating variables

Variables of interest to this problem: \( F, L, S, V, \rho, \mu \) \((n = 6)\)

- Reference dimensions:

\[ [F] = \frac{ML}{T^2} ; \ [L] = L ; \ [S] = L ; \ [V] = \frac{L}{T} ; \ [\mu] = \frac{M}{LT} ; \ [\rho] = \frac{M}{L^3} \]

\((r = 3 \text{ reference dimensions used})\)

- The number of pi terms for this analysis is \( n - r = 6 - 3 = 3 \) pi terms

- Selection of repeating variables: we can choose any variable as a repeating variable. Arbitrarily, let’s pick \( V, L, \rho \) as repeating variables.

- Pi terms determination

The first pi term can be written: \( \Pi_1 = F V^\alpha L^\beta \rho^\gamma \)
In terms of reference dimensions: 
\[ [\Pi_1] = \frac{ML}{T^2} \left( \frac{L}{T} \right)^\alpha L^\beta \left( \frac{M}{L^3} \right)^\gamma = M^0 L^0 T^0 \]

Equation for M: 
\[ 1 + \gamma = 0 \]
Equation for L: 
\[ 1 + \alpha + \beta - 3\gamma = 0 \]
Equation for T: 
\[ -2 - \alpha = 0 \]

Solving for this system:

\[
\begin{align*}
\alpha &= -2 \\
\gamma &= -1 \\
\beta &= -2 
\end{align*}
\]

Therefore:

\[ \Pi_1 = \frac{F}{\rho L^2 V^2} \]

The second pi term can be written: 
\[ \Pi_2 = S V^\alpha L^\beta \rho^\gamma \]

By inspection, we recognize the variable L in the repeating variables shares the same dimensions as S (both are lengths). Therefore, this term can be made dimensionless by taking the ratio of these 2 quantities and setting all other exponents equal to zero.

Therefore, directly:

\[ \Pi_2 = \frac{S}{L} \]

The third pi term can be written: 
\[ \Pi_3 = \mu V^\alpha L^\beta \rho^\gamma \]

By inspection, we recognize the variables used to form the inverse of the Reynolds number.

Therefore, directly:

\[ \Pi_3 = \frac{\mu}{\rho VL} \]

2- Dynamic similarity is achieved by setting each pi term equal between the model and the prototype. In particular, for the third pi term:

\[
\left( \frac{\mu}{\rho VL} \right)_m = \left( \frac{\mu}{\rho VL} \right)_p 
\]

\[ \Rightarrow V_m = V_p \frac{\rho_p L_p \mu_m}{\rho_m L_m \mu_p} \]

The working fluid in the prototype is air: \( \mu_p = 1.81 \times 10^{-5} \text{ kg/m} \cdot \text{s} \); \( \rho_p = 1.21 \text{ kg/m}^3 \)

The working fluid in the model is water: \( \mu_m = 1.01 \times 10^{-3} \text{ kg/m} \cdot \text{s} \); \( \rho_m = 998 \text{ kg/m}^3 \)
Therefore:

\[ V_m = 7.5 \left( \frac{1.21}{998} \right) \left( \frac{10}{1} \right) \left( \frac{1.81 \times 10^{-5}}{1.01 \times 10^{-3}} \right) \Rightarrow V_m = 5.07 \text{ m/s} \]

3. Similarly, matching the first pi term between the model and the prototype:

\[ \left( \frac{F}{\rho L^2 V^2} \right)_m = \left( \frac{F}{\rho L^2 V^2} \right)_p \Rightarrow \frac{F_m}{F_p} = \frac{\rho_m L_m^2 V_m^2}{\rho_p L_p^2 V_p^2} = \left( 998 \right) \left( \frac{1}{10} \right) \left( \frac{1.21}{7.5} \right)^2 \Rightarrow \frac{F_m}{F_p} = 3.77 \]

Problem 3

1. We scale this equation by expressing the initial dimensional equation in terms of scaled variables.

Scaling:

\[
\begin{align*}
y^* &= \frac{y}{h} \\
t^* &= t\omega \\
u^* &= \frac{u}{h\omega}
\end{align*}
\]

The initial equation also includes two derivatives, which need to be expressed in terms of the scaled variables:

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t^*} \frac{\partial}{\partial t^*} = \omega \frac{\partial}{\partial t^*} \\
\frac{\partial}{\partial y} = \frac{\partial}{\partial y^*} \frac{\partial}{\partial y^*} = \frac{1}{h} \frac{\partial}{\partial y^*} \\
\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y^*} \frac{\partial}{\partial y^*} = \frac{1}{h^2} \frac{\partial^2}{\partial y^*^2}
\]

The initial equation can be rewritten:

\[
\rho \frac{\partial u}{\partial t} = X \cos \omega t + \mu \frac{\partial^2 u}{\partial y^2} \Leftrightarrow \rho \omega \frac{\partial (h \omega u^*)}{\partial t^*} = X \cos \left( \frac{t^*}{\omega} \right) + \mu \frac{\partial^2 (h \omega u^*)}{\partial y^*^2}
\]

\[
\Rightarrow \rho \omega^2 h \frac{\partial u^*}{\partial t^*} = X \cos t^* + \frac{\mu \omega}{h} \frac{\partial^2 u^*}{\partial y^*^2}
\]

\[
\Rightarrow \frac{\partial u^*}{\partial t^*} = \frac{X}{\rho \omega^2 h \cos t^*} + \frac{\mu}{\rho \omega h^2} \frac{\partial^2 u^*}{\partial y^*^2}
\]
2. \( h = 1 \text{ cm}, \ \omega = 1 \text{ rad/s}, \ \rho = 1000 \text{ kg/m}^3, \ \mu = 1.12 \times 10^{-3} \text{ kg/m} \cdot \text{s} \) and \( X = 10 \text{ Pa/m} \)

\[
\begin{align*}
\frac{X}{\rho \omega^2 h} &= \frac{10}{1000 \times 1 \times 0.01} = 1 \\
\frac{\mu}{\rho \omega h^2} &= \frac{1.12 \times 10^{-3}}{1000 \times 1 \times 0.0001} = 0.0112
\end{align*}
\]

Therefore, the second coefficient is two order-of-magnitude smaller than the first coefficient. Therefore, in this case, the differential equation to consider reduces to:

\[
\rho \frac{\partial u}{\partial t} = X \cos \omega t
\]