EQUATIONS SUMMARY FOR HIGH REYNOLDS-NUMBER FLOWS

When the boundary layer approximation is employed, the flow is solved using a step-by-step procedure.

Step 1: Outer flow solution
Solve for the outer flow velocity $U(x)$ ignoring the boundary layer. The region of flow outside the boundary layer is approximated as inviscid (and/or irrotational).

Let $U$, $V$ and $P$ be the x-velocity component, y-velocity component and pressure, respectively, in the outer flow region.

The equations of motion in that region are:

$$\begin{align*}
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} &= 0 \\
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} &= -\frac{1}{\rho} \frac{\partial P}{\partial x} \\
U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} &= -\frac{1}{\rho} \frac{\partial P}{\partial y}
\end{align*}$$

Note: those equations are the Euler equations (i.e., Navier Stokes equations without the viscous terms).

Step 2: Boundary layer assumption
Assume that the boundary layer region is thin enough that it does not affect the outer flow solution of step 1.

Step 3: Boundary layer solution
Let $u$, $v$ and $p$ be the x-velocity component, y-velocity component and pressure, respectively, in the inner flow region.
The equations of motion in that region are:

\[
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\frac{u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2} = U \frac{dU}{dx} + \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2} \\
0 &= -\frac{1}{\rho} \frac{\partial p}{\partial y}
\end{aligned}
\]

Notes:
- those equations are the Prandtl's equations (i.e., equations of motion in the boundary layer)
- in this form, this set of equations is valid for the boundary layer on any surface (within the limit of large radius of curvature)

Boundary conditions:
- no-slip at the wall: \( u(y = 0) = 0 \)
- no-penetration at the wall: \( v(y = 0) = 0 \)
- matching condition: \( u \rightarrow U(x) \) as \( y \rightarrow \infty \)
- some known starting profile: \( u = u_0(y) \) at \( x = x_0 \)

**Step 4: Flow characterization**
Calculate flow characteristics of interest (e.g., boundary layer thickness \( \delta(x) \), wall shear stress, etc).

**Step 5: Assumption validation**
Verify that the boundary layer is thin.