

**KUTTA-JOUKOWSKI THEOREM**

**Pressure force on a stationary cylinder**

Consider the flow around the circular cylinder of radius $a$.

The pressure distribution $p_s$ on the surface of the cylinder can be obtained by applying the Bernoulli equation between a point located on the surface of the cylinder and a point located far from the cylinder:

\[
p_0 + \frac{1}{2} \rho U_o^2 + \rho g \bar{z} = p_s + \frac{1}{2} \rho v_{\theta,s}^2 + \rho g \bar{z},
\]

where $v_{\theta,s} = -2U_o \sin \theta$ is the velocity on the cylinder surface (see handout 5.3)

Therefore:

\[
p_s = p_0 + \frac{1}{2} \rho U_o^2 (1 - 4 \sin^2 \theta)
\]

From the knowledge of the local pressure distribution on the cylinder, the resultant pressure force can be calculated along the horizontal and vertical directions:

\[
\begin{align*}
F_x &= -\int_0^{2\pi} p_s \cos \theta (ad \theta) \\
F_y &= -\int_0^{2\pi} p_s \sin \theta (ad \theta)
\end{align*}
\]

$F_x$: force parallel to the uniform stream = drag  
$F_y$: force perpendicular to the uniform stream = lift

Substituting $p_s$ in these expressions yields:

\[
\begin{align*}
F_x &= 0 \\
F_y &= 0
\end{align*}
\]

d’Alembert paradox

d’Alembert paradox: in an irrotational flow, the aerodynamic drag force on a body of any shape immersed in a uniform stream is zero.

**Pressure force on a rotating cylinder**

Bernoulli equation:

\[
p_0 + \frac{1}{2} \rho U_o^2 + \rho g \bar{z} = p_s + \frac{1}{2} \rho v_{\theta,s}^2 + \rho g \bar{z},
\]
where \( v_{\theta,s} = -2U_\infty \sin \theta + \frac{\Gamma}{2\pi a} \) is the velocity on the cylinder surface (see handout 5.4).

Therefore:

\[
p_s = p_0 + \frac{1}{2} \rho U_\infty^2 \left( 1 - 4 \sin^2 \theta + \frac{2\Gamma \sin \theta}{\pi a U_\infty} - \frac{\Gamma^2}{4\pi^2 a^2 U_\infty^2} \right)
\]

Resultant pressure forces:

\[
\begin{cases}
F_x = 0 \\
F_y = -\rho U_\infty \Gamma
\end{cases}
\]

Notes:
- Rotating bodies develop a lift (Magnus effect)
- If \( U_\infty > 0 \) (i.e., along the \(+x\)-direction) and if \( \Gamma > 0 \) (counterclockwise rotation), then \( F_y < 0 \) (downward force)

Kutta-Joukowski theorem: The force acting per unit length on a 2D object of any cross-section is equal to the product of the fluid density, the free-stream velocity and the circulation around any closed contour containing the body (used to determine lift on airfoils).