NAVIER-STOKES AND EULER EQUATIONS

DISCUSSION ON THE EFFECTS OF COMpressibility AND Viscosity

We have derived 2 complete sets of equations for:

1) Newtonian viscous incompressible model
2) Inviscid incompressible model

\textit{In both cases, we ignore compressibility, while in 2), we also ignore viscosity. Under what conditions is this reasonable?}

\textbf{Consider compressibility}

In general, \( \rho = \rho(P, T) \).

Incompressible means:

\[-\frac{1}{\rho} \frac{D\rho}{Dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\]

This requires small changes in \( P \) and \( T \) relative to reference conditions:

- Slowly varying flow
- Small temperature differences between elements of the flow
- Small length scale relative to length scale required for a significant pressure change due to gravity
- \textbf{Low Mach number} (see proof below)

\textbf{Proof}

The Mach number is defined as \( M = \frac{U}{c} \) where \( U \) is a representative speed and \( c \) is the speed of sound.

The requirement of a small change in pressure can be written mathematically as: \( \Delta P / P \ll 1 \).

When this condition is satisfied, \( \Delta P \sim \rho U^2 \).

Additionally, if we consider the case of an ideal gas: \( P = \rho RT \).

Therefore:

\[ \frac{\Delta P}{P} = \frac{\rho U^2}{\rho RT} = \frac{U^2}{RT} = \frac{\gamma U^2}{\gamma RT}, \]

where \( \gamma \) is the ratio of specific heats.

This expression can be rewritten:

\[ \frac{\Delta P}{P} = \frac{\gamma U^2}{c^2} = \gamma M^2. \]

Therefore, \( \Delta P / P \ll 1 \) can also be expressed in terms of the Mach number as: \( \gamma M^2 \ll 1 \).

\textit{Generally,} \( M < 0.3 \) \textit{is used as a working criterion to neglect compressibility.}
Consider viscosity
The classic example is high-speed flow past an airfoil.

We want to compare the order of magnitude of the shear stress $\mu \frac{du}{dy}$ (i.e., a typical viscous term) in the outer and inner flow regions.

The inner flow region is defined by a thickness $\delta$.
In the boundary layer, the characteristic velocity is $u$ and the characteristic dimension in the $y$-direction is $\delta$ ($\delta << 1$).

Therefore, in terms of scales:

$$\mu \frac{du}{dy} \sim \mu \frac{u}{\delta} \quad (1)$$

In the outer region, the characteristic velocity is $u$ and the characteristic dimension in the $y$-direction is the maximum thickness of the airfoil.

Therefore, in terms of scales:

$$\mu \frac{du}{dy} \approx \mu \frac{u}{\text{thickness}} \quad (2)$$

Using (1) and (2):

$$\left. \mu \frac{du}{dy} \right|_{\text{boundary layer}} \sim \left. \mu \frac{u}{\delta} \right|_{\text{outer flow}} = \frac{\text{thickness}}{\delta} \gg 1$$

Therefore, a typical viscous term in the boundary layer is much larger than one in the outer flow. Thus, viscous effects are neglected in the outer flow.