BERNOULLI EQUATION REVISITED

In CV analysis, we derived the Bernoulli equation from the energy equation by writing this equation along a streamline and assuming incompressible, inviscid and steady flow.

The same equation can be derived from the NS equation.

Consider an inviscid flow:

Euler equation:

\[ \frac{D\mathbf{V}}{Dt} = -\nabla P + \rho \mathbf{g} \]

\[ \Rightarrow \rho \frac{\partial \mathbf{V}}{\partial t} + \rho (\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla P + \rho \mathbf{g} \]

(0 (steady))

Therefore, for a steady flow:

\[ \rho (\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla P + \rho \mathbf{g} \]  \hspace{1cm} (i)

Note: \( \mathbf{g} = -g \hat{k} = -g \nabla z \)

In fact: \( \nabla z = \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) z = \frac{\partial z}{\partial x} \hat{i} + \frac{\partial z}{\partial y} \hat{j} + \frac{\partial z}{\partial z} \hat{k} = \hat{k} \)

Substituting back in equation (i):

\[ \rho (\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla P - \rho g \nabla z \]

Using the vector identity \( (\mathbf{V} \cdot \nabla) \mathbf{V} = \frac{1}{2} \nabla (\mathbf{V} \cdot \mathbf{V}) - \mathbf{V} \times (\nabla \times \mathbf{V}) \), the equation can be rewritten:

\[ \frac{1}{2} \nabla (V^2) - \mathbf{V} \times (\nabla \times \mathbf{V}) = -\frac{\nabla P}{\rho} - g \nabla z \]

Projecting this equation along a streamline (local direction vector \( d\mathbf{s} \)):

\[ \frac{1}{2} \nabla (V^2) \cdot d\mathbf{s} - \left[ \mathbf{V} \times (\nabla \times \mathbf{V}) \right] \cdot d\mathbf{s} = -\frac{\nabla P}{\rho} \cdot d\mathbf{s} - g \nabla z \cdot d\mathbf{s} \]  \hspace{1cm} (ii)

- Let’s look at the first term of equation (ii):
\[
\frac{1}{2} \nabla (V^2) \cdot \text{d}s = \frac{1}{2} \left( \frac{\partial (V^2)}{\partial x} \hat{i} + \frac{\partial (V^2)}{\partial y} \hat{j} + \frac{\partial (V^2)}{\partial z} \hat{k} \right) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k})
\]

Therefore:
\[
\frac{1}{2} \nabla (V^2) \cdot \text{d}s = \frac{1}{2} \left( \frac{\partial (V^2)}{\partial x} \, dx + \frac{\partial (V^2)}{\partial y} \, dy + \frac{\partial (V^2)}{\partial z} \, dz \right) = \frac{1}{2} d (V^2) \quad (iii)
\]

Similarly, the two terms on the right-hand side of equation (ii) can be rewritten:

\[
\frac{\nabla P}{\rho} \cdot \text{d}s = \frac{1}{\rho} \, dP \quad (iv)
\]

\[
g \nabla z \cdot \text{d}s = g dz \quad (v)
\]

- Let's look at the second term of equation (ii):
\[
\left[ \nabla \times (\nabla \times \text{V}) \right] \cdot \text{d}s = 0 \quad (vi)
\]

Substituting (iii), (iv), (v) and (vi) into equation (ii):

\[
\frac{1}{2} d (V^2) = - \frac{dP}{\rho} - g dz
\]

Following integration between 2 points located along the same streamline:

\[
\frac{1}{2} V^2 + \frac{P}{\rho} + g z = \text{constant}
\]

This equation is the Bernoulli equation and is only valid along a streamline for steady, incompressible, inviscid flow.