PRESSURE GRADIENT IN CYLINDRICAL COORDINATES

Let's consider a cylindrical fluid wedge of angle $\Delta \theta$ and dimensions $\Delta r$ and $\Delta z$ along the radial and axial directions, respectively.
We will assume a pressure $p$ at the center ($r = \Delta r/2 ; \theta = 0 ; z = 0$) of the wedge.

Each face of the wedge is exposed to a pressure force. The resultant pressure force along the radial direction is $F_r = F_n - F_z$

$$F_n = \left[ p - \frac{\partial p}{\partial r} \frac{\Delta r}{2} + O(\Delta r^2) \right] \left( r - \frac{\Delta r}{2} \right) \Delta \theta \Delta z$$

$$F_z = \left[ p + \frac{\partial p}{\partial r} \frac{\Delta r}{2} + O(\Delta r^2) \right] \left( r + \frac{\Delta r}{2} \right) \Delta \theta \Delta z$$

Therefore:

$$F_r = -\frac{\partial p}{\partial r} r \Delta r \Delta \theta \Delta z + O \left( r \Delta r^2 \Delta \theta \Delta z \right)$$
The resultant pressure force along the tangential direction is $F_\theta = F_{\theta_1} - F_{\theta_2}$

\[
F_{\theta_1} = \left[ p - \frac{\partial p}{\partial \theta} \frac{\Delta \theta}{2} + O\left(\Delta \theta^2\right) \right] \Delta r \Delta z
\]
\[
F_{\theta_2} = \left[ p + \frac{\partial p}{\partial \theta} \frac{\Delta \theta}{2} + O\left(\Delta \theta^2\right) \right] \Delta r \Delta z
\]

Therefore:
\[
F_\theta = -\frac{\partial p}{\partial \theta} \Delta r \Delta \theta \Delta z + O\left(\Delta r \Delta \theta^2 \Delta z\right)
\]

Finally, the resultant pressure force along the axial direction is $F_z = F_{z_1} - F_{z_2}$

\[
F_{z_1} = \left[ p - \frac{\partial p}{\partial z} \frac{\Delta z}{2} + O\left(\Delta z^2\right) \right] r \Delta \theta \Delta r
\]
\[
F_{z_2} = \left[ p + \frac{\partial p}{\partial z} \frac{\Delta z}{2} + O\left(\Delta z^2\right) \right] r \Delta \theta \Delta r
\]
Therefore: 

\[ F_z = -\frac{\partial p}{\partial z} r \Delta r \Delta \theta \Delta z + O \left( r \Delta r \Delta \theta \Delta z^2 \right) \]

The resultant pressure force on the fluid wedge is: 

\[ \Delta F = \Delta F_\rho \hat{e}_\rho + \Delta F_\theta \hat{e}_\theta + \Delta F_z \hat{e}_z \]

It follows that: 

\[ \frac{\Delta F}{\Delta \mathbf{V}} = \frac{\Delta F_\rho}{\Delta \mathbf{V}} \hat{e}_\rho + \frac{\Delta F_\theta}{\Delta \mathbf{V}} \hat{e}_\theta + \frac{\Delta F_z}{\Delta \mathbf{V}} \hat{e}_z \]

which can be written:

\[ \frac{\Delta F}{\Delta \mathbf{V}} = \frac{\Delta F}{r \Delta r \Delta \theta \Delta z} = \left[ -\frac{\partial p}{\partial r} + O(\Delta r) \right] \hat{e}_r + \left[ -\frac{1}{r} \frac{\partial p}{\partial \theta} + O \left( \frac{1}{\Delta \theta} \right) \right] \hat{e}_\theta + \left[ -\frac{\partial p}{\partial z} + O(\Delta z) \right] \hat{e}_z \]

The gradient of the pressure can be calculated as:

\[ -\nabla p = \lim_{\Delta \mathbf{V} \to 0} \frac{\Delta F}{\Delta \mathbf{V}} = -\frac{\partial p}{\partial r} \hat{e}_r - \frac{1}{r} \frac{\partial p}{\partial \theta} \hat{e}_\theta - \frac{\partial p}{\partial z} \hat{e}_z \]