Problem 1

1- Process representation

Process 1→2: polytropic process: \( P\mathcal{V} = C \). This function can be plotted in the \( P - \mathcal{V} \) diagram as: \( P = C/\mathcal{V}^n \).

Process 2→3: isobaric expansion: \( \mathcal{V} \rightarrow \) as \( P \) remains constant. This process is shown as a straight horizontal line on the \( P - \mathcal{V} \) diagram.

2- Compression-to-expansion work ratio

The work during the polytropic compression is calculated as:

\[
W_{12} = \int_1^2 P d\mathcal{V} = \int_1^2 \frac{C}{\mathcal{V}^n} d\mathcal{V},
\]

which can be integrated as:

\[
W_{12} = \frac{C}{1-n} [\mathcal{V}^{1-n}]_1^2
\]

\[
\Rightarrow W_{12} = \frac{C(\mathcal{V}_2^{1-n} - \mathcal{V}_1^{1-n})}{1-n}
\]

Since \( C = P_1\mathcal{V}_1^n = P_2\mathcal{V}_2^n \):

\[
W_{12} = \frac{P_2\mathcal{V}_2^{1-n} - P_1\mathcal{V}_1^{1-n}}{1-n}
\]

Using the IG law ( \( P\mathcal{V} = mRT \)):

\[
W_{12} = \frac{mR}{1-n} (T_2 - T_1)
\]
Based on the diagram representation, this work is negative (during compression, the gas receives work). Therefore, the magnitude of the work during the compression process is:

\[
W_{\text{compression}} = -W_{12} = \frac{mR}{n-1} (T_2 - T_1)
\]

The work during the isobaric expansion is calculated as:

\[
W_{23} = \int_{2}^{3} P d\mathcal{V} = P(\mathcal{V}_3 - \mathcal{V}_2) = P_3 \mathcal{V}_3 - P_2 \mathcal{V}_2
\]

Using the IG law \( P\mathcal{V} = mRT \):

\[
W_{23} = W_{\text{expansion}} = mR(T_3 - T_2)
\]

Therefore:

\[
r_w = \frac{W_{\text{compression}}}{W_{\text{expansion}}} = \frac{1}{n-1} \frac{T_2 - T_1}{T_3 - T_2} = \frac{1}{n-1} \frac{T_2 - T_1}{T_3 / T_2 - 1}
\]

Since the problem requires to obtain an expression including the compression ratio, we need to consider again each process.

During the polytropic process:

\[
P_1 \mathcal{V}_1^n = P_2 \mathcal{V}_2^n \Rightarrow mRT_1 \mathcal{V}_1^{n-1} = mRT_2 \mathcal{V}_2^{n-1} \Rightarrow \frac{T_1}{T_2} = \left( \frac{\mathcal{V}_2}{\mathcal{V}_1} \right)^{n-1} = \left( \frac{1}{r} \right)^{n-1} = r^{1-n}
\]

During the isobaric process:

\[
P = \text{constant} \Rightarrow \frac{mRT}{\mathcal{V}} = \text{constant} \Rightarrow \frac{T}{\mathcal{V}} = \text{constant}
\]

Therefore:

\[
\frac{T_3}{T_2} = \frac{\mathcal{V}_2}{\mathcal{V}_1} = \frac{\mathcal{V}_2}{\mathcal{V}_2} = r
\]

Substituting into the expression for \( r_w \):

\[
r_w = \frac{W_{\text{compression}}}{W_{\text{expansion}}} = \frac{1}{n-1} \frac{1 - r^{1-n}}{r-1}
\]

3- Numerical application

\[
r_w = - \frac{1}{1.4 - 1} \frac{1 - 6^{1-1.4}}{6-1} = \frac{r_w}{0.256}
\]
Problem 2

1- 1st law of thermodynamics, mass of ice calculation

The system consists of the water and the ice. This is a closed system (constant mass and volume throughout the process). Assuming no change in kinetic and potential energy, the 1st law can be formulated as follows:

\[ U_2 - U_1 = Q_{12} - W_{12} \]

- Since the volume of the system is constant throughout the process: \( d\Omega = 0 \Rightarrow W_{12} = 0 \)
- Since the glass is adiabatic: \( Q_{12} = 0 \)
- The change in internal energy of the system is due to the change in internal energy of the water and the change in internal energy of the ice.

Therefore, the 1st law reduces to:

\[ (U_2 - U_1)_w + (U_2 - U_1)_i = 0 \]

Assuming that water is modeled as an incompressible substance:

\[ (U_2 - U_1)_w = m_w c_w (T_2 - T_{1,w}) \]

where \( T_{1,w} \) is the initial water temperature.

The change in internal energy of the ice during the entire process can be split into three quantities:

1. The change in internal energy of the ice in solid form from the initial ice temperature \( T_{1,i} \) to the melting temperature \( T_{melt} \)
2. The change in internal energy of ice in liquid form from the melting temperature \( T_{melt} \) to the final equilibrium temperature \( T_2 \)
3. The increase in internal energy due to the change in phase from solid to liquid. This change is expressed as the heat of fusion: \( h_f \)

Therefore:

\[ (U_2 - U_1)_i = m_i c_i (T_{melt} - T_{1,i}) + m_i c_w (T_2 - T_{melt}) + m_i h_f \]

The first law can then be rewritten as:

\[ m_w c_w (T_2 - T_{1,w}) + m_i c_i (T_{melt} - T_{1,i}) + m_i c_w (T_2 - T_{melt}) + m_i h_f = 0 \]

And the mass of ice is expressed as:

\[ m_i = m_w \frac{c_w (T_{1,w} - T_2)}{c_i (T_{melt} - T_{1,i}) + c_w (T_2 - T_{melt}) + h_f} \Rightarrow m_i = \rho_d \frac{c_w (T_{1,w} - T_2)}{c_i (T_{melt} - T_{1,i}) + c_w (T_2 - T_{melt}) + h_f} \]
2- Numerical application with ice at 0°C and -20°C

Ice at 0°C: We use Table A-3 to obtain the properties of ice.

**Table A-3**

<table>
<thead>
<tr>
<th>Properties of common liquids, solids, and foods (Concluded)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b) Solids (values are for room temperature unless indicated otherwise)</td>
</tr>
<tr>
<td>Substance</td>
</tr>
<tr>
<td>Metals</td>
</tr>
<tr>
<td>Bronze (76% Cu, 2% Zn, 2% Al)</td>
</tr>
<tr>
<td>Brass, yellow (65% Cu, 35% Zn)</td>
</tr>
<tr>
<td>Copper</td>
</tr>
<tr>
<td>-173°C</td>
</tr>
<tr>
<td>-100°C</td>
</tr>
<tr>
<td>-50°C</td>
</tr>
<tr>
<td>0°C</td>
</tr>
<tr>
<td>Nonmetals</td>
</tr>
<tr>
<td>Graphite</td>
</tr>
<tr>
<td>Granite</td>
</tr>
<tr>
<td>Gypsum or plaster board</td>
</tr>
<tr>
<td>Ice</td>
</tr>
<tr>
<td>220 K</td>
</tr>
<tr>
<td>240 K</td>
</tr>
<tr>
<td>260 K</td>
</tr>
<tr>
<td>273 K</td>
</tr>
</tbody>
</table>

⇒ $c_i = 2.11 \text{ kJ/kg} \cdot \text{K}$

For water at 20°C, we also use Table A-3 to obtain the thermal properties:

**Table A-3**

<table>
<thead>
<tr>
<th>Properties of common liquids, solids, and foods</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Liquids</td>
</tr>
<tr>
<td>Substance</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Water</td>
</tr>
<tr>
<td></td>
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<td></td>
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<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

⇒ $c_w \approx 4.18 \text{ kJ/kg} \cdot \text{K}$

**Numerical application:**

$$m_i = (1 \times 0.3) \times \frac{4.18 \times (20 - 5)}{2.11 \times (0 - 0) + 4.18 \times (5 - 0) + 333.7} \Rightarrow m_{i,0^\circ C} = 53.1 \text{ g}$$

If the ice is at -20°C, the specific heat becomes $c_i = 1.94 \text{ kJ/kg} \cdot \text{K}$ (Table A-3).

**Numerical application:**

$$m_i = (1 \times 0.3) \times \frac{4.18 \times (20 - 5)}{1.94 \times (0 + 20) + 4.18 \times (5 - 0) + 333.7} \Rightarrow m_{i,-20^\circ C} = 47.8 \text{ g}$$
3- **Numerical application with water at 0°C**

If cold water is used instead of ice, the expression becomes:

\[ m_{\text{cold, w}} = m_w \frac{c_w(T_{1,w} - T_2)}{c_{\text{cold, w}}(T_2 - T_{\text{cold, w}})} \]

We use table A-3 to obtain the properties of water at 0°C:

<table>
<thead>
<tr>
<th>TABLE A–3</th>
<th>Properties of common liquids, solids, and foods</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Liquids</td>
<td>Boiling data at 1 atm</td>
</tr>
<tr>
<td>Substance</td>
<td>Normal boiling point, °C</td>
</tr>
<tr>
<td>Water</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>75</td>
</tr>
<tr>
<td></td>
<td>100</td>
</tr>
</tbody>
</table>

\[ c_{\text{cold, w}} = 4.22 \text{ kJ/kg·K} \]

**Numerical application:**

\[ m_{\text{cold, w}} = \frac{(1 \times 0.3) \times 4.18 \times (20 - 5)}{4.22 \times (5 - 0)} \implies m_{\text{cold, w}} = 891 \text{ g} \]