

Name:

Equations:

$$\lim_{n \rightarrow \infty} \left( a_n \frac{u_n}{u_{n+1}} - a_{n+1} \right) = C = \begin{cases} > 0 & \text{convergent,} \\ < 0 & \text{divergent,} \\ = 0 & \text{fails.} \end{cases} \quad (\text{Kummer}), \quad (1)$$

$$\lim_{n \rightarrow \infty} n \left( \frac{u_n}{u_{n+1}} - 1 \right) = P = \begin{cases} > 1 & \text{convergent,} \\ < 1 & \text{divergent,} \end{cases} \quad (\text{Raabe}), \quad (2)$$

$$\frac{u_n}{u_{n+1}} = 1 + \frac{h}{n} + \frac{B(n)}{n^2} \quad (\text{Gauss}), \quad (3)$$

$$\nabla \psi(q_i) = \mathbf{e}_1 \frac{1}{h_1} \frac{\partial \psi}{\partial q_1} + \mathbf{e}_2 \frac{1}{h_2} \frac{\partial \psi}{\partial q_2} + \mathbf{e}_3 \frac{1}{h_3} \frac{\partial \psi}{\partial q_3}, \quad (4)$$

$$\nabla \cdot \mathbf{V}(q_i) = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} (V_1 h_2 h_3) + \frac{\partial}{\partial q_2} (V_2 h_3 h_1) + \frac{\partial}{\partial q_3} (V_3 h_1 h_2) \right], \quad (5)$$

$$\nabla \times \mathbf{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \mathbf{e}_1 h_1 & \mathbf{e}_2 h_2 & \mathbf{e}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}, \quad (6)$$

$$\nabla^2 = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left( \frac{h_1 h_3}{h_2} \frac{\partial}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial}{\partial q_3} \right) \right] \quad (7)$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \quad (\text{Ampere's law}) \quad (8)$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho \quad (\text{Gauss' law}) \quad (9)$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (\text{Faraday's law}) \quad (10)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{Gauss' law}) \quad (11)$$

1. (a) Discuss the convergence of the infinite series

$$\sum_n \frac{(n-r)!}{n!}$$

for various integer values of  $r$ . [10]

- (b) Find the radius of convergence of [5]

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{n}\right)^{n^2} z^n \quad (p \in \mathcal{R}).$$

- (c) In a quantum theory of a system of oscillators, the average energy of the system is given by [5]

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} nh\nu e^{-nx}}{\sum_{n=0}^{\infty} e^{-nx}},$$

where  $x = h\nu/(k_B T)$ . Evaluate  $\langle E \rangle$ .

2. (a) From Maxwell's equations in free space, show that the electric field satisfies the vector wave equation [10]

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

- (b) Derive  $\nabla^2 \mathbf{V}$  in cylindrical polar coordinates. [10]

3. (a) Find an analytic function of  $z$  whose imaginary part is  $(y \cos y + x \sin y)e^x$ . [5]

- (b) i. For the function

$$f(z) = \ln \left( \frac{z+c}{z-c} \right),$$

where  $c$  is real, show that the real part  $u$  of  $f$  is constant on a circle of radius  $c \operatorname{cosech} u$  centered on the point  $z = c \coth u$ . [10]

- ii. Use this result to show that the electrical capacity per unit length of two parallel cylinders of radii  $a$ , placed with their axes  $2d$  apart, is proportional to  $(\cosh^{-1}(d/a))^{-1}$ . [5]

4. (a) Write down the first 3 terms of the Laurent expansion of  $\sin z/z^4$  about  $z = 0$ . [5]

- (b) Calculate the residue of  $\sin z/z^4$ . [5]

- (c) Evaluate the following integral: [10]

$$\int_0^{\infty} dt \frac{t \sin \alpha t}{1+t^2}, \quad \alpha > 0.$$

## Sample Exam Solution 1

1. (a)

$$\sum_n \frac{(n-r)!}{n!}$$

- $r = 0$ :

$$\lim_{n \rightarrow \infty} a_n \neq 0,$$

: divergent.

- $r = 1$ : harmonic series — divergent.
- $r = 2$ :

$$\sum_2^{\infty} \frac{(n-2)!}{n!} = \sum_2^{\infty} \left[ \frac{1}{n-1} - \frac{1}{n} \right] = \left[ \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots \right] = 1,$$

convergent.

- $r > 2$ : Since

$$\frac{(n-2)!}{n!} > \frac{(n-r)!}{n!},$$

by comparison test, convergent.

(b)

$$\sum_{n=1}^{\infty} \left( \frac{n+p}{n} \right)^{n^2} z^n \quad (p \in \mathcal{R}).$$

$$\lim_{n \rightarrow \infty} (a_n)^{1/n} = \lim_{n \rightarrow \infty} \left[ \left( \frac{n+p}{n} \right)^{n^2} \right]^{1/n} = \lim_{n \rightarrow \infty} \left( \frac{n+p}{n} \right)^n = \lim_{n \rightarrow \infty} \left( 1 + \frac{p}{n} \right)^n = e^p.$$

Thus,

$$R = \lim_{n \rightarrow \infty} \frac{1}{(a_n)^{1/n}} = e^{-p}.$$

(c)

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} nh\nu e^{-nx}}{\sum_{n=0}^{\infty} e^{-nx}} = -h\nu \frac{\partial}{\partial x} \ln \sum_{n=0}^{\infty} e^{-nx}.$$

Now

$$\sum_{n=0}^{\infty} e^{-nx} = \sum_{n=0}^{\infty} [e^{-x}]^n = \frac{1}{1 - e^{-x}}.$$

Therefore,

$$\langle E \rangle = -h\nu \frac{\partial}{\partial x} \ln \frac{1}{1 - e^{-x}} = h\nu \frac{\partial}{\partial x} \ln(1 - e^{-x}) = \frac{h\nu}{e^x - 1}.$$

2. (a) Taking the curl of Eq. (10) and using Eq. (8):

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{E}) &= -\frac{\partial \nabla \times \mathbf{B}}{\partial t} = -\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \\ \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.\end{aligned}$$

But, by Eq. (9),

$$\nabla \cdot \mathbf{E} = 0,$$

and

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

(b) For this, we will need the following results:

$$\hat{\rho} = \hat{x} \cos \phi + \hat{y} \sin \phi, \quad (12)$$

$$\hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi, \quad (13)$$

$$\hat{z} = \hat{z}, \quad (14)$$

$$\frac{\partial \hat{\rho}}{\partial \phi} = \hat{\phi}, \quad \frac{\partial \hat{\phi}}{\partial \phi} = -\hat{\rho}. \quad (15)$$

$$\begin{aligned}\nabla^2 \mathbf{V} &= \nabla^2 (V_\rho \hat{\rho} + V_\phi \hat{\phi} + V_z \hat{z}) \\ &= \nabla^2 (V_\rho \hat{\rho} + V_\phi \hat{\phi}) + (\nabla^2 V_z) \hat{z}.\end{aligned}$$

Now,

$$\begin{aligned}\nabla^2 (V_\rho \hat{\rho}) &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} (V_\rho \hat{\rho}) \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} (V_\rho \hat{\rho}) + \frac{\partial^2}{\partial z^2} (V_\rho \hat{\rho}) \\ &= \frac{\hat{\rho}}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V_\rho}{\partial \rho} \right) + \hat{\rho} \frac{1}{\rho^2} \frac{\partial^2 V_\rho}{\partial \phi^2} + \hat{\rho} \frac{\partial^2 V_\rho}{\partial z^2} + \frac{1}{\rho^2} \left[ V_\rho \frac{\partial^2 \hat{\rho}}{\partial \phi^2} + 2 \frac{\partial V_\rho}{\partial \phi} \frac{\partial \hat{\rho}}{\partial \phi} \right] \\ &= \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V_\rho}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V_\rho}{\partial \phi^2} + \frac{\partial^2 V_\rho}{\partial z^2} \right] \hat{\rho} - \frac{V_\rho}{\rho^2} \hat{\rho} + \frac{2}{\rho^2} \frac{\partial V_\rho}{\partial \phi} \hat{\phi} \\ &= \nabla^2 V_\rho \hat{\rho} - \frac{V_\rho}{\rho^2} \hat{\rho} + \frac{2}{\rho^2} \frac{\partial V_\rho}{\partial \phi} \hat{\phi}, \quad (16)\end{aligned}$$

$$\begin{aligned}\nabla^2 (V_\phi \hat{\phi}) &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} (V_\phi \hat{\phi}) \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} (V_\phi \hat{\phi}) + \frac{\partial^2}{\partial z^2} (V_\phi \hat{\phi}) \\ &= \nabla^2 V_\phi \hat{\phi} - \frac{V_\phi}{\rho^2} \hat{\phi} - \frac{2}{\rho^2} \frac{\partial V_\phi}{\partial \phi} \hat{\rho}, \quad (17)\end{aligned}$$

$$\nabla^2 \mathbf{V} = \left[ \nabla^2 V_\rho - \frac{V_\rho}{\rho^2} - \frac{2}{\rho^2} \frac{\partial V_\rho}{\partial \phi} \right] \hat{\rho} + \left[ \nabla^2 V_\phi - \frac{V_\phi}{\rho^2} + \frac{2}{\rho^2} \frac{\partial V_\phi}{\partial \phi} \right] \hat{\phi} + (\nabla^2 V_z) \hat{z} \quad (18)$$

3. (a)

$$\begin{aligned} \operatorname{Im}f(z) &= v = (y \cos y + x \sin y)e^x, \\ \frac{\partial v}{\partial y} &= (\cos y + x \cos y)e^x - y \sin ye^x = \frac{\partial u}{\partial x}, \\ u &= \int dx [(\cos y + x \cos y)e^x - y \sin ye^x] = x \cos ye^x - y \sin ye^x + g(y), \\ \frac{\partial u}{\partial y} &= -x \sin ye^x - \sin ye^x - y \cos ye^x + g'. \end{aligned}$$

But

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -(y \cos y + x \sin y)e^x - \sin ye^x.$$

Therefore,

$$g(y) = 0,$$

and

$$\begin{aligned} f &= u + iv = (x \cos y - y \sin y)e^x + i(y \cos y + x \sin y)e^x \\ &= (xe^{iy} + iye^{iy})e^x = ze^z. \end{aligned}$$

(b) i.

$$\begin{aligned} f(z) &= \ln \left( \frac{z+c}{z-c} \right) = \ln \left( \left| \frac{z+c}{z-c} \right| e^{i\theta} \right), \\ u &= \ln \left| \frac{z+c}{z-c} \right|, \\ |z+c| &= e^u |z-c|. \end{aligned}$$

Expanding,

$$\begin{aligned} (1 - e^{2u})|z|^2 + (1 - e^{2u})c^2 + 2c(1 + e^{2u})\operatorname{Re}z &= 0, \\ \sinh u|z|^2 + \sinh uc^2 - 2c \cosh u \operatorname{Re}z &= 0, \\ \sinh^2 u|z|^2 + \sinh^2 uc^2 - 2c \cosh u \sinh u \operatorname{Re}z &= 0, \\ \sinh^2 u|z|^2 + \cosh^2 uc^2 - 2c \cosh u \sinh u \operatorname{Re}z &= c^2, \\ |z|^2 + \coth^2 uc^2 - 2c \coth u \operatorname{Re}z &= \frac{c^2}{\sinh^2 u}, \\ |z - c \coth u| &= c \operatorname{cosech} u. \end{aligned}$$

ii. If the centers of the two cylinders are chosen such that

$$c \coth u_1 = -d, c \coth u_2 = d,$$

then, they are  $2d$  apart. Both radii are

$$a = |c \operatorname{cosech} u_i|.$$

These equations are consistent with

$$\cosh u_i = d/a, \quad u_1 = -u_2.$$

Then, the capacity is

$$C = \frac{Q}{\Delta V} \propto \frac{1}{u_2 - u_1} = \frac{1}{2u} \propto \frac{1}{u_i} \propto \frac{1}{\cosh^{-1}(d/a)}.$$

4. (a)

$$\frac{\sin z}{z^4} = \frac{1}{z^4} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) = \left( \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \dots \right).$$

(b)  $a_{-1} = -\frac{1}{3!}$ .

(c)

$$\int_0^\infty dt \frac{t \sin \alpha t}{1+t^2} = \frac{1}{2} \operatorname{Im} \int_{-\infty}^\infty dt \frac{te^{i\alpha t}}{1+t^2}.$$

Use Jordan's lemma to close in UHP. Only pole is at  $t = i$  with residue

$$a_{-1} = \frac{1}{2} e^{-\alpha},$$

and

$$\oint dt \frac{te^{i\alpha t}}{1+t^2} = 2\pi i \frac{1}{2} e^{-\alpha} = i\pi e^{-\alpha},$$

giving

$$\int_0^\infty dt \frac{t \sin \alpha t}{1+t^2} = \frac{\pi}{2} e^{-\alpha}.$$