Homework 9: Partial Differential Equations — Separation of Variables

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1. For a homogeneous spherical solid with constant thermal diffusivity K, and no heat sources, the equation of heat conduction becomes

$$\frac{\partial u(r,t)}{\partial t} = K\nabla^2 u(r,t)\,.$$

Assuming a solution of the form

$$u(r,t) = R(r)T(t)\,,$$

   (a) show that the radial equation may take on the form [5]

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + 2r\frac{\mathrm{d}R}{\mathrm{d}r} + [\alpha^2r^2 - l(l+1)]R = 0\,; \tag{1}$$

   (b) by setting

$$\alpha r = x,\;\; R(r) = \frac{Z(x)}{x^{1/2}}\,,$$

   in Eq. (1), find the equation for $Z(x)$. What is the name of the resulting equation? [2]

2. Separate the heat diffusion equation in cylindrical polar coordinates. [5]

3. Consider the 1D time-independent Schrödinger equation with the potential

$$V(x) = \frac{1}{2}kx^2\,.$$

   (a) Using $\xi = ax$ and

$$a = \left(\frac{mk}{\hbar^2}\right)^{1/4},\;\; \lambda = \frac{2E}{\hbar}\left(\frac{m}{k}\right)^{1/2},$$

   show that [2]

$$\frac{\mathrm{d}^2\psi(\xi)}{\mathrm{d}\xi^2} + (\lambda - \xi^2)\psi(\xi) = 0\,.$$

   (b) Using the substitution

$$\psi(\xi) = y(\xi)e^{-\xi^2/2}\,,$$

   find the new equation satisfied by y [2]. What is the name of this equation? [1]

4. The parabolic rotational coordinate sytem is defined as follows:

$$
\begin{aligned}
x &= \xi\,\eta\cos\phi, \\
y &= \xi\,\eta\sin\phi, \\
z &= \frac{1}{2}(\eta^2 - \xi^2), \\
0 &\leq \xi, \eta < \infty, 0 < \phi < 2\pi.
\end{aligned}
$$

Surfaces of constant coordinates are

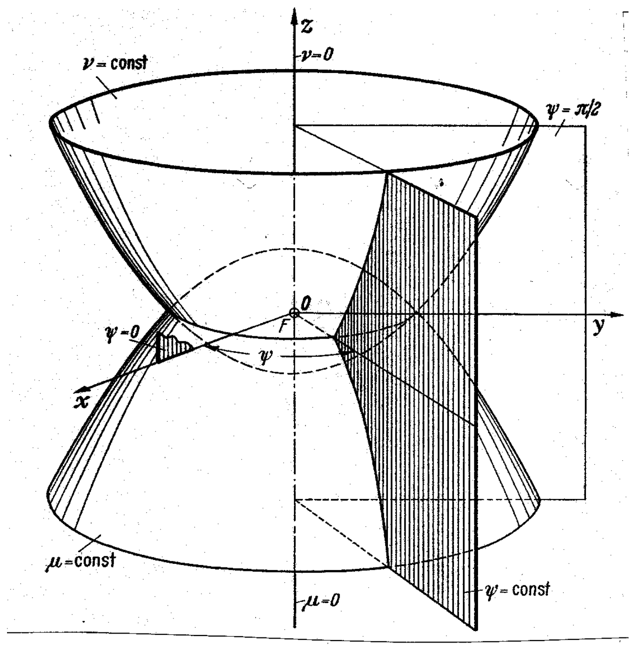

Figure 1: Parabolic rotational coordinates.

$$
\begin{aligned}
\xi &= \text{constant: paraboloid, , } +z \text{ axis,} \\
\eta &= \text{constant: paraboloid, , } -z \text{ axis,} \\
\phi &= \text{constant: half-planes.}
\end{aligned}
$$

(a) Show that the metric is given by [5]

$$
\begin{aligned}
h_\xi &= h_\eta = (\eta^2 + \xi^2)^{1/2}, \\
h_\phi &= \eta\xi.
\end{aligned}
$$

(b) Show that the Laplacian is [5]

$$\nabla^2 = \frac{1}{(\xi^2+\eta^2)}\left[\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right)\right] + \frac{1}{\xi^2\eta^2}\frac{\partial^2}{\partial\phi^2}.$$

(c) The pressure field of an acoustic wave satisfies the Helmholtz equation,

$$\nabla^2 f(\mathbf{r}) + k^2 f(\mathbf{r}) = 0,$$

where k is a constant known as the wave number and $f(\mathbf{r})$ is the eigenfunction, inside a domain V bounded by a set of intersecting surfaces S_i and with appropriate boundary conditions on the surfaces. Writing

$$f(\xi,\eta,\phi) = M(\xi)N(\eta)\Phi(\phi),$$

show that one gets the following separated ordinary differential equations: [5]

$$\begin{aligned} \frac{d^2\Phi}{d\phi^2} + k_3^2\Phi &= 0, \\ \frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dM}{d\xi}\right) + \left[k_2 + k^2\xi^2 - \frac{k_3^2}{\xi^2}\right]M &= 0, \\ \frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{dN}{d\eta}\right) - \left[k_2 - k^2\eta^2 + \frac{k_3^2}{\eta^2}\right]N &= 0, \end{aligned}$$

where k_2 and k_3 are separation constants.

Solution

1. The Laplacian in spherical polar is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,.$$

If the solution has no angular dependence, then

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\,.$$

(a) Using

$$u(r,t) = R(r)T(t)\,,$$

the diffusion equation becomes

$$\begin{aligned} \frac{K}{r^2R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) &= \frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} \equiv -\alpha^2\,, \\ T(t) &= e^{-\alpha^2 t}\,, \end{aligned}$$

and

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + 2r\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{\alpha^2}{K}r^2R = 0\,.$$

(b) If

$$\alpha r = x, R(r) = \frac{Z(x)}{x^{1/2}},$$

then

$$x^2Z'' + xZ' + [x^2 - (1/2)^2]Z = 0,$$

and

$$R(\alpha r) = \frac{Z_{1/2}(\alpha r)}{(\alpha r)^{1/2}}.$$

The new equation is Bessel's equation.

2. The Laplacian in cylindrical polar coordinates is given by

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\,,$$

and the solution can be written as

$$\psi(\mathbf{r}) = R(\rho)\Phi(\phi)Z(z)T(t)\,.$$

Then, the diffusion equation becomes

$$\frac{\Phi ZT}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) + \frac{RZT}{\rho^2}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} + R\Phi T\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = \frac{R\Phi Z}{K}\frac{\mathrm{d}T}{\mathrm{d}t}\,,$$

or

$$\frac{1}{R\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) + \frac{1}{\rho^2\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} + \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = \frac{1}{KT}\frac{\mathrm{d}T}{\mathrm{d}t} \equiv -\alpha\,.$$

Next, if

$$\frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = \beta^2\,,$$

then

$$\frac{\rho}{R}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) + (\alpha+\beta^2)\rho^2 = -\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} \equiv m^2\,.$$

Finally

$$\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) + [(\alpha+\beta^2)\rho^2 - m^2]R = 0\,.$$

3. Schrödinger's equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \frac{1}{2}kx^2\psi(x) = E\psi(x)\,.$$

(a) Using $\xi = ax$, Schrödinger's equation becomes

$$-\frac{a^2\hbar^2}{2m}\frac{\mathrm{d}^2\psi(\xi)}{\mathrm{d}\xi^2} + \frac{1}{2}\frac{k}{a^2}\xi^2\psi(\xi) = E\psi(\xi)\,,$$

or

$$-\frac{a^4\hbar^2}{mk}\frac{\mathrm{d}^2\psi(\xi)}{\mathrm{d}\xi^2} + \xi^2\psi(\xi) = \frac{2a^2}{k}E\psi(\xi)\,.$$

If

$$a = \left(\frac{mk}{\hbar^2}\right)^{1/4}\,,\ \lambda = \frac{2E}{\hbar}\left(\frac{m}{k}\right)^{1/2}\,,$$

then

$$\frac{\mathrm{d}^2\psi(\xi)}{\mathrm{d}\xi^2} + (\lambda - \xi^2)\psi(\xi) = 0\,. \tag{2}$$

(b) Using the substitution,

$$\psi(\xi) = y(\xi)e^{-\xi^2/2}\,,$$

then

$$\psi' = y'e^{-\xi^2/2} - \xi y e^{-\xi^2/2},\ \psi'' = y''e^{-\xi^2/2} - 2\xi y'e^{-\xi^2/2} - ye^{-\xi^2/2} + \xi^2 ye^{-\xi^2/2},$$

and Eq. (2) becomes

$$y'' - 2\xi y' + (\lambda - 1)y = 0\,.$$

This is the Hermite equation.

4. (a) Metric:

$$h_i = g_{ii}^{1/2} = \sqrt{\frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_i} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_i} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_i}}\,.$$

Then

$$h_\xi = h_\eta = \left[\eta^2\cos^2\phi + \eta^2\sin^2\phi + \xi^2\right]^{1/2} = (\eta^2 + \xi^2)^{1/2}, \qquad (3)$$

$$h_\phi = \left[\xi^2\eta^2\sin^2\phi + \xi^2\eta^2\cos^2\phi\right]^{1/2} = \eta\xi. \qquad (4)$$

(b) Laplacian: Using

$$\nabla^2 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial q_3}\right)\right],$$

one obtains

$$\nabla^2 = \frac{1}{(\xi^2+\eta^2)}\left[\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right)\right] + \frac{1}{\xi^2\eta^2}\frac{\partial^2}{\partial\phi^2}. \qquad (5)$$

(c) Writing

$$f(\xi,\eta,\phi) = M(\xi)N(\eta)\Phi(\phi),$$

then

$$\nabla^2 f(\mathbf{r}) + k^2 f(\mathbf{r})$$
$$= \frac{1}{(\xi^2+\eta^2)}\left[\frac{N\Phi}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}M}{\mathrm{d}\xi}\right) + \frac{M\Phi}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}N}{\mathrm{d}\eta}\right)\right] + \frac{MN}{\xi^2\eta^2}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} + k^2MN\Phi = 0,$$

and

$$\frac{\xi^2\eta^2}{(\xi^2+\eta^2)}\left[\frac{1}{\xi M}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}M}{\mathrm{d}\xi}\right) + \frac{1}{\eta N}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}N}{\mathrm{d}\eta}\right)\right] + k^2\xi^2\eta^2 = -\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} \equiv k_3^2\,,$$

$$\left[\frac{1}{\xi M}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}M}{\mathrm{d}\xi}\right) + \frac{1}{\eta N}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}N}{\mathrm{d}\eta}\right)\right] + k^2(\xi^2+\eta^2) - k_3^2\frac{(\xi^2+\eta^2)}{\xi^2\eta^2} = 0\,.$$

Rewriting the latter gives

$$\left[\frac{1}{\xi M}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}M}{\mathrm{d}\xi}\right)+k^2\xi^2-\frac{k_3^2}{\xi^2}\right]=-\left[\frac{1}{\eta N}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}N}{\mathrm{d}\eta}\right)+k^2\eta^2-\frac{k_3^2}{\eta^2}\right]\equiv -k_2\,.$$

Finally,

$$\begin{aligned}
\frac{d^2\Phi}{d\phi^2}+k_3^2\Phi &= 0,\\
\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dM}{d\xi}\right)+\left[k_2+k^2\xi^2-\frac{k_3^2}{\xi^2}\right]M &= 0,\\
\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{dN}{d\eta}\right)-\left[k_2-k^2\eta^2+\frac{k_3^2}{\eta^2}\right]N &= 0,
\end{aligned}$$

where k_2 and k_3 are separation constants.