Homework 8: Bessel Function

1. Given Bessel's differential equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

   (a) Show that the roots of the indicial equation are $\pm\nu$. [2]

   (b) Show that the recurrence relation is [2]

$$a_n(\sigma) = -\frac{1}{(n+\sigma)^2 - \nu^2} a_{n-2}.$$

   (c) By computing the series solutions, show that the Bessel functions of order $\nu = \pm 1/2$ are [6]

$$x^{-1/2} \sin x, x^{-1/2} \cos x.$$

2. Show that

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-)^s}{s!(s+v)!} \left(\frac{x}{2}\right)^{v+2s}$$

   satisfies [4]

$$\begin{aligned} J_{v-1} + J_{v+1} &= \frac{2v}{x} J_v\,, \\ J_{v-1} - J_{v+1} &= 2J_v' \end{aligned}$$

   and, by direct differentiation, Bessel's differential equation [2]

$$x^2 J_\nu'' + xJ_\nu' + (x^2 - \nu^2)J_\nu = 0.$$

3. Using only the generating function

$$g(x,t) = e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n,$$

   show that $J_n(x)$ has fixed parity for integral n: [4]

$$J_n(x) = (-)^n J_n(-x).$$