

Homework 7: Series Solution

1. Using the transformation $x = 1/z$, show that

$$y'' + P(x)y' + Q(x)y = 0,$$

becomes

$$\frac{d^2y}{dz^2} + \left(\frac{2}{z} - \frac{P(z^{-1})}{z^2} \right) \frac{dy}{dz} + \frac{Q(z^{-1})}{z^4}y = 0.$$

2. For the equation

$$y'' - 2xy' - 2y = 0,$$

- (a) calculate the two series solutions at $x = 0$; [2.5]
 (b) show that one solution is e^{x^2} ; [2.5]
 (c) given the closed form solution, attempt to find the second solution using [2.5]

$$y_2(x) = f(x)y_1(x).$$

Hence, obtain the following series expansion of the error function: [2.5]

$$\int_0^x du e^{-u^2} = e^{-x^2} \sum_{n=0}^{\infty} \frac{2^n x^{2n+1}}{(2n+1)!!}.$$

One can fix the constants by comparing coefficients for small x .

3. Write the first three terms in the series solutions of [10]

- (a)

$$\frac{d}{d\eta} \left[(1 - \eta^2) \frac{du}{d\eta} \right] + \alpha u + \beta \eta^2 u = 0.$$

- (b) (do only for indicial root = 1)

$$\frac{d^2\psi}{dx^2} + \left(E' - \frac{A'e^{-ax}}{x} \right) \psi = 0.$$

Solution

1.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \frac{dz}{dx} = -z^2 \frac{dy}{dz}, \\ \frac{d^2y}{dx^2} &= z^2 \frac{d}{dz} \left(z^2 \frac{dy}{dz} \right).\end{aligned}$$

2. (a)

$$\begin{aligned}\Theta(\sigma) &= \sigma(\sigma - 1) + \sigma p_0 + q_0 = \sigma(\sigma - 1) = 0, \\ \sigma &= 0, 1, \\ a_n &= -\frac{\sum_{r=0}^{n-1} a_r [(\sigma + r)p_{n-r} + q_{n-r}]}{\Theta(\sigma + n)} = \frac{2}{\sigma + n} a_{n-2}.\end{aligned}$$

For $\sigma = 0$, $a_{2n} = \frac{1}{n!} a_0$, and

$$y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}.$$

For $\sigma = 1$, $a_{2n} = \frac{2^n}{(2n+1)!!} a_0$, and

$$y_2(x) = a_0 x \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!!} x^{2n}.$$

(b)

$$y_1(x) = e^{x^2}.$$

(c) Let $y(x) = f(x)e^{x^2}$. Then

$$\begin{aligned}y' &= f'e^{x^2} + 2xf'e^{x^2}, \\ y'' &= f''e^{x^2} + 4xf'e^{x^2} + 2fe^{x^2} + 4x^2fe^{x^2}.\end{aligned}$$

This gives

$$\begin{aligned}f'' + 2xf' &= 0, \\ (f')' + 2x(f') &= 0, \\ f' &= e^{-x^2}, \\ f(x) &= A \int_0^x du e^{-u^2} + B, \\ y(x) &= e^{x^2} \left[A \int_0^x du e^{-u^2} + B \right] \\ &= a_0 \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!!} x^{2n+1}.\end{aligned}$$

If $x = 0 \implies B = 0$. For small x ,

$$\begin{aligned} LHS &= \int_0^\infty du [1 - u^2] = x - \frac{1}{3}x^3, \\ RHS &= e^{-x^2} \sum_{n=0}^\infty \frac{2^n}{(2n+1)!!} x^{2n+1} \approx (1 - x^2)x \approx x, \\ A &= 1. \end{aligned}$$

3. (a) Let $u(\eta) = \sum_{m=0}^\infty \eta^{m+k}$.

$$\begin{aligned} &\sum_{m=0}^\infty a_m \left[(m+k)(m+k-1)\eta^{m+k-2} + \{\alpha - (m+k)(m+k+1)\}\eta^{m+k} \right. \\ &\left. + \beta\eta^{m+k+2} \right] = 0. \end{aligned}$$

Gives

$$\eta^m : a_0 k(k-1) = 0 \implies k = 0, \text{ or } 1 \text{ (exclude } a_0 = 0 \text{)}.$$

If $k = 0$,

$$\begin{aligned} m = 1 & : a_1 k(k+1) = 0, \\ m = 2 & : a_2(k+1)(k+2) + a_0[\alpha - k(k+1)] = 0, \\ m = 3 & : a_3(k+2)(k+3) + a_1[\alpha - (k+1)(k+2)] = 0, \end{aligned}$$

$$m = 4 : a_{m+4}(m+k+3)(m+k+4) + a_{m+2}[\alpha - (m+k+2)(m+k+3)] + \beta a_m = 0.$$

Thus,

$$\begin{aligned} a_2 &= -a_0 \frac{\alpha}{2}, \\ a_4 &= \frac{1}{12} \left[\frac{\alpha(\alpha-6)}{2} - \beta \right] a_0, \\ a_{m+4} &= -\frac{1}{(m+3)(m+4)} \left\{ [\alpha - (m+2)(m+3)] a_{m+2} + \beta a_m \right\}, \\ &\dots \\ u(\eta) &= a_0 \left\{ 1 - \frac{\alpha}{2}\eta^2 + \frac{1}{12} \left[\frac{\alpha(\alpha-6)}{2} - \beta \right] \eta^4 + \dots \right\}. \end{aligned}$$

If $k = 1$,

$$\begin{aligned} m = 1 & : a_1 k(k+1) = 0 \implies a_1 = 0, \\ m = 2 & : a_2(k+1)(k+2) + a_0[\alpha - k(k+1)] = 0, \\ m = 3 & : a_3(k+2)(k+3) + a_1[\alpha - (k+1)(k+2)] = 0, \end{aligned}$$

$$m = 4 : a_{m+4}(m+k+3)(m+k+4) + a_{m+2}[\alpha - (m+k+2)(m+k+3)] + \beta a_m = 0.$$

Thus,

$$\begin{aligned} a_2 &= a_0 \frac{(2-\alpha)}{6}, \\ a_4 &= \left[\frac{(2-\alpha)(12-\alpha)}{120} - \frac{\beta}{20} \right] a_0, \\ &\dots \\ u(\eta) &= a_0 \eta \left\{ 1 + \frac{(2-\alpha)}{6} \eta^2 + \left[\frac{(2-\alpha)(12-\alpha)}{120} - \frac{\beta}{20} \right] \eta^4 + \dots \right\}. \end{aligned}$$

(b) Let $\psi = \sum_m a_m x^{m+k}$. Then,

$$\sum_m a_m \left[(m+k)(m+k-1)x^{m+k-2} + E'x^{m+k} - A'x^{m+k-1}e^{-ax} \right] = 0.$$

Expanding the exponential

$$\begin{aligned} a_0 k(k-1) &= 0 \quad \implies k = 1, \\ a_1 k(k+1) - a_0 A' &= 0 \quad \implies a_1 = \frac{a_0}{2} A', \\ a_2(k+1)(k+2) - a_1 A' + (E' + aA')a_0 &= 0, \\ a_2 &= \frac{a_0}{6} \left\{ \frac{A'^2}{2} - (E' + aA') \right\}, \\ \psi(x) &= a_0 x \left[1 + \frac{A'}{2} x + \frac{1}{6} \left\{ \frac{A'^2}{2} - (E' + aA') \right\} x^2 + \dots \right]. \end{aligned}$$