Homework Solution 2

1. (i)

$$\nabla \cdot (\mathbf{r} f(r)) \stackrel{\text{ith}}{=} \partial_i (x_i f(r)) = x_i (\partial_i f) + f \partial_i x_i.$$

But

$$\partial_i x_i = 3,$$

and

$$\begin{aligned} \frac{\partial}{\partial x_i} &= \frac{\partial r}{\partial x_i}\frac{\partial}{\partial r} = \frac{x_i}{r}\frac{\partial}{\partial r}, \\ x_i \frac{\partial}{\partial x_i} &= \frac{x_i x_i}{r}\frac{\partial}{\partial r} = r\frac{\partial}{\partial r}. \end{aligned}$$

Then,

$$\nabla \cdot (\mathbf{r} f(r)) = 3f(r) + r\frac{\mathrm{d}f}{\mathrm{d}r}. \tag{1}$$

(ii) Using $f(r) = r^{n-1}$ in Eq. (1):

$$\nabla \cdot (\mathbf{r} r^{n-1}) = 3r^{n-1} + r(n-1)r^{n-2} = (n+2)r^{n-1}.$$

(iii)

$$\begin{aligned} \mathbf{A} \times (\nabla \times \mathbf{B}) &= \varepsilon_{ijk} A_j (\nabla \times \mathbf{B})_k = \varepsilon_{ijk}\varepsilon_{klm} A_j \partial_l B_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j \partial_l B_m = A_j \partial_i B_j - A_j \partial_j B_i \\ &= \nabla_B (\mathbf{A} \cdot \mathbf{B}) - (\mathbf{A} \cdot \nabla)\mathbf{B}. \end{aligned}$$

(iv)

$$\begin{aligned} \nabla \times (f\mathbf{V}) &= \varepsilon_{ijk}\partial_j (f V_k) = \varepsilon_{ijk}(\partial_j f) V_k + \varepsilon_{ijk} f \partial_j V_k && (2) \\ &= f\nabla \times \mathbf{V} + \nabla f \times \mathbf{V}. && (3) \end{aligned}$$

(v) Using $\mathbf{V} = \mathbf{r}$ in Eq. (3):

$$\begin{aligned} \nabla \times \mathbf{r} &= \varepsilon_{ijk}\partial_j x_k = \varepsilon_{ijk}\delta_{jk} = 0, \\ \nabla f &= \partial_i f = \frac{x_i}{r} f' = \widehat{\mathbf{r}} f', \\ \nabla \times (\mathbf{r} f(r)) &= \nabla f \times \mathbf{r} + f \nabla \times \mathbf{r} = 0. \end{aligned}$$

2. Given a central force,
$$\mathbf{F} = f(r)\hat{\mathbf{r}},$$
then
$$\begin{aligned}
\mathbf{L} &= \mathbf{r} \times \mathbf{p}, \\
\tau &= \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \hat{\mathbf{r}} f(r) = 0, \\
\mathbf{L} &= \text{constant.}
\end{aligned}$$

3. Using an infinitesimally thin pillbox and integrating Maxwell's equation over the volume,
$$\int \mathrm{d}V\, \nabla \cdot \mathbf{D} = \int \mathrm{d}V\, \rho,$$
one can convert to surface integrals (using Gauss's law for the LHS) and neglect the side contributions to give the result.

4. (a)
$$\nabla \times \left[\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right] = \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0,$$
by Maxwell's equation. Thus,
$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi.$$

(b)
$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A})\right].$$
Using
$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}),$$
then
$$\mathbf{v} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{A} \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{v} - \mathbf{A} \times (\nabla \times \mathbf{v}).$$
But $\mathbf{v}$ is the velocity of the particle (i.e., it is not a field); hence, the last two terms above are zero and
$$\begin{aligned}
\mathbf{F} &= q\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{A} \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla)\mathbf{A}\right] \\
&= q\left[-\nabla\phi - \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} + \nabla(\mathbf{A} \cdot \mathbf{v})\right].
\end{aligned}$$

5. $\nabla\psi$

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\underline{\widehat{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\underline{\widehat{\phi}} + \frac{\partial\psi}{\partial z}\underline{\widehat{z}}\,. \tag{4}$$

$\nabla\cdot\mathbf{V}$

$$\nabla\cdot\mathbf{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho V_\rho\right) + \frac{1}{\rho}\frac{\partial V_\phi}{\partial\phi} + \frac{\partial V_z}{\partial z}\,. \tag{5}$$

∇^2

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\,. \tag{6}$$

$\nabla^2\mathbf{V}$

For this, we will need the following results:

$$\begin{aligned}
\underline{\widehat{\rho}} &= \underline{\widehat{x}}\cos\phi + \underline{\widehat{y}}\sin\phi, && (7)\\
\underline{\widehat{\phi}} &= -\underline{\widehat{x}}\sin\phi + \underline{\widehat{y}}\cos\phi, && (8)\\
\underline{\widehat{z}} &= \underline{\widehat{z}}, && (9)\\
\frac{\partial\underline{\widehat{\rho}}}{\partial\phi} &= \underline{\widehat{\phi}},\ \frac{\partial\underline{\widehat{\phi}}}{\partial\phi} = -\underline{\widehat{\rho}}. && (10)
\end{aligned}$$

$$\begin{aligned}
\nabla^2\mathbf{V} &= \nabla^2(V_\rho\underline{\widehat{\rho}} + V_\phi\underline{\widehat{\phi}} + V_z\underline{\widehat{z}})\\
&= \nabla^2(V_\rho\underline{\widehat{\rho}} + V_\phi\underline{\widehat{\phi}}) + (\nabla^2 V_z)\underline{\widehat{z}}.
\end{aligned}$$

Now,

$$\begin{aligned}
\nabla^2(V_\rho\underline{\widehat{\rho}}) &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\left(V_\rho\underline{\widehat{\rho}}\right)\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\left(V_\rho\underline{\widehat{\rho}}\right) + \frac{\partial^2}{\partial z^2}\left(V_\rho\underline{\widehat{\rho}}\right)\\
&= \frac{\underline{\widehat{\rho}}}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V_\rho}{\partial\rho}\right) + \underline{\widehat{\rho}}\frac{1}{\rho^2}\frac{\partial^2 V_\rho}{\partial\phi^2} + \underline{\widehat{\rho}}\frac{\partial^2 V_\rho}{\partial z^2} + \frac{1}{\rho^2}\left[V_\rho\frac{\partial^2\underline{\widehat{\rho}}}{\partial\phi^2} + 2\frac{\partial V_\rho}{\partial\phi}\frac{\partial\underline{\widehat{\rho}}}{\partial\phi}\right]\\
&= \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V_\rho}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V_\rho}{\partial\phi^2} + \frac{\partial^2 V_\rho}{\partial z^2}\right]\underline{\widehat{\rho}} - \frac{V_\rho}{\rho^2}\underline{\widehat{\rho}} + \frac{2}{\rho^2}\frac{\partial V_\rho}{\partial\phi}\underline{\widehat{\phi}}\\
&= \nabla^2 V_\rho\underline{\widehat{\rho}} - \frac{V_\rho}{\rho^2}\underline{\widehat{\rho}} + \frac{2}{\rho^2}\frac{\partial V_\rho}{\partial\phi}\underline{\widehat{\phi}}, && (11)
\end{aligned}$$

$$\nabla^2(V_\phi \underline{\widehat{\phi}}) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\left(V_\phi\underline{\widehat{\phi}}\right)\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\left(V_\phi\underline{\widehat{\phi}}\right) + \frac{\partial^2}{\partial z^2}\left(V_\phi\underline{\widehat{\phi}}\right)$$

$$= \nabla^2 V_\phi\underline{\widehat{\phi}} - \frac{V_\phi}{\rho^2}\underline{\widehat{\phi}} - \frac{2}{\rho^2}\frac{\partial V_\phi}{\partial\phi}\underline{\widehat{\rho}}, \qquad (12)$$

$$\nabla^2\mathbf{V} = \left[\nabla^2 V_\rho - \frac{V_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial V_\phi}{\partial\phi}\right]\underline{\widehat{\rho}} + \left[\nabla^2 V_\phi - \frac{V_\phi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial V_\rho}{\partial\phi}\right]\underline{\widehat{\phi}} + (\nabla^2 V_z)\underline{\widehat{z}}\,. \qquad (13)$$