Homework Solution 1

1. (i) divergent: compare with $1/n$.
$$\frac{1}{\ln n} > \frac{1}{n}.$$
(ii) divergent: ratio.
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{n+1}{10} \to n.$$
(iii) convergent: Raabe.
$$R_n = n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{2(n+1)(2n+3)}{2n(2n+1)} - 1\right) = n\frac{8n+3}{2n(2n+1)} \to 2.$$
(iv) convergent $\forall\, p > 1$: Cauchy-Maclaurin. With $f(x) = x^{-p}$,
$$\int_1^\infty \mathrm{d}x\, x^{-p} = \begin{cases} \left.\frac{x^{-p+1}}{-p+1}\right|_1^\infty & p \neq 1, \\ \left.\ln x\right|_1^\infty & p = 1, \end{cases}$$
Therefore, the series is divergent $\forall\, p \leq 1$.

(v) divergent: Gauss.
$$\frac{u_{2j}}{u_{2j+2}} = \frac{(2j+1)(2j+2)}{2j(j+1) - l(l+1)} \overset{j>>l}{\to} \frac{(2j+1)(2j+2)}{2j(j+1)} = 1 + \frac{1}{j}.$$
Hence, divergent since $h = 1$.

2. (i) $\forall\, 1 \leq x < \infty$ (Abel's test).
From 1(iv), $\zeta(x)$ is convergent for $x > 1$; thus, the alternating series is convergent for $x > 1$. Furthermore, the Leibniz criterion for alternating series implies that the alternating harmonic series ($x = 1$) is also convergent. Using Abel's test with $f(x) = 1$, the series is uniformly convergent for $1 \leq x < \infty$.

(ii) $\forall\, 1 < s \leq x < \infty$ (Abel + Riemann zeta).
Similarly to previous except for $x = 1$ which is the harmonic series and no longer convergent.

3. (a)
$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{\rho}\frac{\mathrm{d}m}{\mathrm{d}t} = 2 \times 10^4\,\mathrm{m}^3/\mathrm{yr}.$$

(b)

$$\Delta r = \left(\frac{3}{4\pi}\right)^{1/3} \left[V^{1/3}(t+T) - V^{1/3}(t)\right] \approx 0.$$

(c)

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x).$$

(d)

$$\delta f \approx h f'(x).$$

(e) The remainder in the Taylor expansion is

$$R_n = \frac{h^n}{n!} f^{(n)}(\xi).$$

Here

$$\begin{aligned} r(V) &= \left(\frac{3V}{4\pi}\right)^{1/3}, \\ R_n &= \frac{1}{n!}\binom{\frac{1}{3}}{n}\left(\frac{\delta V}{V}\right)^n V^{1/3} \overset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Note that the series here is an alternating one and

$$\frac{|a_{n+1}|}{|a_n|} \to \frac{\delta V}{V}$$

for large n. Hence, by the Leibniz criterion, the series is absolutely convergent.

(f)

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}r} &= 4\pi r^2 = \frac{3V}{r}, \\ \frac{\delta r}{\delta t} &= \frac{r}{3V}\frac{\delta V}{\delta t}. \end{aligned}$$

(g) Accretion rate $\approx 1\,\text{Å/yr}$.