Homework 13: Series Solution II

1. Show that for any equation for which the origin is an ordinary point, the indicial equation has roots 0 and 1. [5]

2. For the equation
$$x^2y'' - \frac{3}{2}xy' + (1+x)y = 0\,,$$
find the (i) recurrence relation, [2] and (ii) general term of the series with $\sigma = 1/2$. [2] Show that the series converges for all finite x. [1]

3. For the equation
$$y'' - 2xy' - 2y = 0,$$
   (a) calculate the two series solutions; [2.5]
   (b) show that one solution is e^{x^2}; [2.5]
   (c) given the closed form solution, attempt to find the second solution using [2.5]
$$y_2(x) = f(x)y_1(x).$$
   Hence, obtain the following series expansion of the error function: [2.5]
$$\int_0^x \mathrm{d}u\, e^{-u^2} = e^{-x^2}\sum_{n=0}^{\infty}\frac{2^n x^{2n+1}}{(2n+1)!!}.$$

4. Find the series solution of [10]
   (a)
$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[(1-\eta^2)\frac{\mathrm{d}u}{\mathrm{d}\eta}\right] + \alpha u + \beta\eta^2 u = 0.$$
   (b)
$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + (E' - \frac{A'e^{-ax}}{x})\psi = 0 \qquad (A < 0).$$

Solution

1.

$$\begin{aligned}\Theta(\sigma) &= \sigma(\sigma-1)+\sigma p_0+q_0=\sigma(\sigma-1)=0,\\ \sigma &= 0,1\,.\end{aligned}$$

2.

$$y(x)=x^{1/2}+2x^{3/2}-x^{1/2}\sum_{n=2}^{\infty}\frac{(-2x)^n}{n!(2n-3)!!}\,.$$

3. (a)

$$\begin{aligned}\Theta(\sigma) &= \sigma(\sigma-1)+\sigma p_0+q_0=\sigma(\sigma-1)=0,\\ \sigma &= 0,1,\\ a_n &= -\frac{\sum_{r=0}^{n-1}a_r\left[(\sigma+r)p_{n-r}+q_{n-r}\right]}{\Theta(\sigma+n)}=\frac{2}{\sigma+n}a_{n-2}.\end{aligned}$$

For $\sigma=0, a_{2n}=\frac{1}{n!}a_0$, and

$$y_1(x)=a_0\sum_{n=0}^{\infty}\frac{1}{n!}x^{2n}\,.$$

For $\sigma=1, a_{2n}=\frac{2^n}{(2n+1)!!}a_0$, and

$$y_2(x)=a_0x\sum_{n=0}^{\infty}\frac{2^n}{(2n+1)!!}x^{2n}.$$

(b)

$$y_1(x)=e^{x^2}.$$

(c) Let $y(x)=f(x)e^{x^2}$. Then

$$\begin{aligned}y' &= f'e^{x^2}+2xfe^{x^2},\\ y'' &= f''e^{x^2}+4xf'e^{x^2}+2fe^{x^2}+4x^2fe^{x^2}.\end{aligned}$$

This gives

$$\begin{aligned} f'' + 2xf' &= 0, \\ (f')' + 2x(f') &= 0, \\ f' &= e^{-x^2}, \\ f(x) &= A\int_0^x \mathrm{d}u\, e^{-u^2} + B, \\ y(x) &= e^{x^2}\left[A\int_0^x \mathrm{d}u\, e^{-u^2} + B\right] \\ &= a_0 \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!!} x^{2n+1}. \end{aligned}$$

If $x = 0 \Longrightarrow B = 0$. For small x,

$$\begin{aligned} LHS &= \int_0^{\infty} \mathrm{d}u\,[1-u^2] = x - \frac{1}{3}x^3, \\ RHS &= e^{-x^2}\sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!!} x^{2n+1} \approx (1-x^2)x \approx x, \\ A &= 1. \end{aligned}$$

4. (a) Let $u(\eta) = \sum_{m=0}^{\infty} \eta^{m+k}$.

$$\sum_{m=0}^{\infty} a_m \Big[(m+k)(m+k-1)\eta^{m+k-2} + \{\alpha - (m+k)(m+k+1)\}\,\eta^{m+k} + \beta\eta^{m+k+2}\Big] = 0.$$

Gives

$$\begin{aligned} \eta^m &: \; a_0 k(k-1) = 0 \qquad \Longrightarrow k = 1 \text{ (exclude } a_0 = k = 0), \\ m = 1 &: \; a_1 k(k+1) = 0 \qquad \Longrightarrow a_1 = 0, \\ m = 2 &: \; a_2(k+1)(k+2) + a_0[\alpha - k(k+1)] = 0, \\ m = 3 &: \; a_3(k+2)(k+3) + a_1[\alpha - (k+1)(k+2)] = 0, \end{aligned}$$

$m = 4 : a_{m+4}(m+k+3)(m+k+4) + a_{m+2}[\alpha - (m+k+2)(m+k+3)] + \beta a_m = 0.$

Thus,

$$\begin{aligned} a_2 &= a_0 \frac{(2-\alpha)}{6}, \\ a_4 &= \left[\frac{(2-\alpha)(12-\alpha)}{120} - \frac{\beta}{20}\right] a_0, \\ &\cdots \\ u(\eta) &= a_0\eta\left[1 + \frac{(2-\alpha)}{6}\eta^2 + \left[\frac{(2-\alpha)(12-\alpha)}{120} - \frac{\beta}{20}\right]\eta^4 + \cdots\right]. \end{aligned}$$

(b) Let $\psi = \sum_m a_m x^{m+k}$. Then,

$$\sum_m a_m \left[(m+k)(m+k-1)x^{m+k-2} + E' x^{m+k} - A' x^{m+k-1} e^{-ax}\right] = 0.$$

Expanding the exponential

$$\begin{aligned}
& a_0 k(k-1) = 0 \qquad \Longrightarrow k = 1, \\
& a_1 k(k+1) - a_0 A' = 0 \qquad \Longrightarrow a_1 = \frac{a_0}{2} A', \\
& a_2 (k+1)(k+2) - a_1 A' + (E' + aA') a_0 = 0, \\
& a_2 = \frac{a_0}{6} \left\{ \frac{A'^2}{2} - (E' + aA') \right\}, \\
\psi(x) \;\; = \;\; & a_0 x \left[1 + \frac{A'}{2} x + \frac{1}{6} \left\{ \frac{A'^2}{2} - (E' + aA') \right\} x^2 + \cdots \right].
\end{aligned}$$