PHY673 Mathematical Physics I (F10): Exam 1

Time: 8:30–9:30

No. questions: 3 (60 points)

Name:

Equations:

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, \tag{1}$$

$$h_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2}, \tag{2}$$

$$\nabla\psi(q_i) = \mathbf{e}_1 \frac{1}{h_1}\frac{\partial \psi}{\partial q_1} + \mathbf{e}_2 \frac{1}{h_2}\frac{\partial \psi}{\partial q_2} + \mathbf{e}_3 \frac{1}{h_3}\frac{\partial \psi}{\partial q_3}\,, \tag{3}$$

$$\nabla\cdot\mathbf{V}(q_i) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(V_2 h_3 h_1) + \frac{\partial}{\partial q_3}(V_3 h_1 h_2)\right], \tag{4}$$

$$\nabla\times\mathbf{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \mathbf{e}_1 h_1 & \mathbf{e}_2 h_2 & \mathbf{e}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}, \tag{5}$$

$$\oint_S \mathbf{V}\cdot \mathrm{d}\mathbf{S} = \int_V \nabla\cdot\mathbf{V}\,\mathrm{d}V\,, \tag{6}$$

$$\oint_\Gamma \mathbf{V}\cdot \mathrm{d}\mathbf{l} = \int_S (\nabla\times\mathbf{V})\cdot \mathrm{d}\mathbf{S}\,, \tag{7}$$

$$\int_V (u\nabla^2 v - v\nabla^2 u)\,\mathrm{d}V = \oint_S (u\nabla v - v\nabla u)\cdot \mathrm{d}\mathbf{S}. \tag{8}$$

1. (a) Show that [5]

$$\nabla \times (f\mathbf{V}) = f\nabla \times \mathbf{V} + \nabla f \times \mathbf{V}.$$

   (b) Evaluate $\nabla \times \mathbf{r}$. [5]

   (c) Using Gauss' theorem or otherwise, show that [5]

$$\oint_S \mathrm{d}\mathbf{S} \times \mathbf{P} = \int_V (\nabla \times \mathbf{P})\, \mathrm{d}V\,.$$

   (d) The field of a magnetic dipole of moment $\mathbf{M}$, placed in a vacuum at the origin, is given by $\mathbf{H} = -\nabla\Omega$, where Ω is the magnetostatic potential $\Omega = -(1/4\pi)\mathbf{M} \cdot \nabla(1/r)$. Show that [5]

$$\mathbf{H} = (1/4\pi)[3\hat{\mathbf{r}}(\mathbf{M} \cdot \mathbf{r})r^{-4} - \mathbf{M}r^{-3}]\,.$$

2. The 2D parabolic coordinates (μ, ν) are defined in terms of the 2D cartesian as follows:

$$x = \frac{1}{2}\left(\mu^2 - \nu^2\right), y = \mu\nu,$$

and

$$0 \leq \mu < \infty, -\infty < \nu < +\infty.$$

   (a) Show that the scale factors are [5]

$$h_\mu = h_\nu = (\mu^2 + \nu^2)^{1/2}.$$

   (b) Obtain an expression for the gradient operator. [5]

   (c) Obtain an expression for the divergence of a vector. [5]

   (d) Obtain an expression for the Laplacian operator. [5]

3. (a) Give the Laplacian in spherical polar coordinates. [5]

   (b) Separate the Laplace equation in spherical polar coordinates giving the resulting ordinary differential equations. [10]

   (c) If the separation constant in the radial equation is written as $l(l+1)$, show that the general solution to the radial equation can be written as [5]

$$R(r) = Ar^l + Br^{-(l+1)}.$$

Exam Solution 1

1. (a) Using Gauss' theorem

$$\oint_S \mathbf{V}\cdot d\mathbf{S} = \int_V \nabla\cdot\mathbf{V}\, dV\,,$$

with $\mathbf{V} = \mathbf{P}\times\mathbf{a}$, and $\mathbf{a}$ being a constant vector,

$$\begin{aligned}
\oint_S \mathbf{P}\times\mathbf{a}\cdot d\mathbf{S} &= \int_V \nabla\cdot(\mathbf{P}\times\mathbf{a})\, dV \\
\mathbf{a}\cdot\oint_S d\mathbf{S}\times\mathbf{P} &= \mathbf{a}\cdot\int_V (\nabla\times\mathbf{P})\, dV \\
\oint_S d\mathbf{S}\times\mathbf{P} &= \int_V (\nabla\times\mathbf{P})\, dV\,.
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla\times(f\mathbf{V}) &= \varepsilon_{ijk}\partial_j(fV_k) = \varepsilon_{ijk}(\partial_j f)V_k + \varepsilon_{ijk} f\partial_j V_k \\
&= f\nabla\times\mathbf{V} + \nabla f\times\mathbf{V}.
\end{aligned}$$

(c)

$$\nabla\times\mathbf{r}_i = \varepsilon_{ijk}\partial_j x_k = \varepsilon_{ijk}\delta_{jk} = 0\,.$$

(d)

$$\begin{aligned}
\mathbf{H} &= -\nabla\Omega = \frac{1}{4\pi}\nabla\left(\mathbf{M}\cdot\nabla\frac{1}{r}\right) \\
&= \frac{1}{4\pi}\partial_i\left(M_j\partial_j\left(\frac{1}{r}\right)\right) = \frac{1}{4\pi}M_j\partial_i\partial_j\left(\frac{1}{r}\right) \\
&= \frac{1}{4\pi}M_j\partial_i\left(-\frac{1}{r^2}\frac{x_j}{r}\right) = -\frac{1}{4\pi}M_j\left[\frac{1}{r^3}\partial_i x_j - \frac{3}{r^4}\frac{x_j x_i}{r}\right] \\
&= (1/4\pi)[3\hat{\mathbf{r}}(\mathbf{M}\cdot\mathbf{r})r^{-4} - \mathbf{M}r^{-3}]\,.
\end{aligned}$$

2. (a) Use

$$h_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2}.$$

(b)

$$\nabla = \frac{\mathbf{e}_\mu}{(\mu^2+\nu^2)^{1/2}}\frac{\partial}{\partial\mu} + \frac{\mathbf{e}_\nu}{(\mu^2+\nu^2)^{1/2}}\frac{\partial}{\partial\nu}.$$

(c)

$$\nabla\cdot\mathbf{V} = \frac{1}{(\mu^2+\nu^2)}\left\{\frac{\partial}{\partial\mu}\left[(\mu^2+\nu^2)^{1/2}V_\mu\right] + \frac{\partial}{\partial\nu}\left[(\mu^2+\nu^2)^{1/2}V_\nu\right]\right\}.$$

(d)

$$\nabla^2 = \frac{1}{(\mu^2+\nu^2)}\left[\frac{\partial^2}{\partial\mu^2}+\frac{\partial^2}{\partial\nu^2}\right].$$

3. (a) Since,

$$h_r = 1, h_\theta = r, h_\phi = r\sin\theta$$

then

$$\nabla^2 = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right].$$

(b) The Laplace equation is

$$\nabla^2\psi = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi = 0.$$

Let

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi).$$

Then

$$\begin{aligned}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}+k_3^2\Phi &= 0,\\ \frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\left[k_2^2-\frac{k_3^2}{\sin^2\theta}\right]\Theta &= 0,\\ \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)-k_2^2R &= 0.\end{aligned}$$

(c) Writing $k_2^2 = l(l+1)$,

$$R(r) = Ar^l + Br^{-(l+1)}.$$