On the derivation of Coriolis and other noninertial accelerations

Guy Vandegrift
Department of Physics, University of Texas at El Paso, El Paso, Texas 79968-0515[a)]

(Received 6 October 1994; accepted 11 January 1995)

I finally found a convincing derivation of the Coriolis and other noninertial terms that arise when a particle's acceleration is observed from a reference frame undergoing arbitrary motion. My treatment is standard, but makes a subtle change in notation which has the pedagogical advantage of allowing postponement of the concept of time derivatives in "fixed" and "rotating" frames until the end.[1] Textbooks generally introduce two time derivatives ("fixed" and "rotating"), which takes a bit of thought to understand. Whenever possible, such definitions should be avoided as part of an important derivation.

We start from a fixed coordinate system with unit vectors $\mathbf{x}_i$ $(i=1,2,3)$ and consider a moving particle represented by the vector $\boldsymbol{\rho}=\boldsymbol{\rho}(t)$. Let $\mathbf{R}=\mathbf{R}(t)$ be the origin of noninertial coordinate system, and let $\mathbf{r}=\mathbf{r}(t)$ be the particle's position relative to the moving frame, so that $\boldsymbol{\rho}=\mathbf{R}+\mathbf{r}$.

The three orthogonal unit vectors of the rotating frame are $\mathbf{e}_i=\mathbf{e}_i(t)$. Textbooks give an adequate derivation of

$$\dot{\hat{\mathbf{e}}}_i=\boldsymbol{\omega}\times\hat{\mathbf{e}}_i \tag{1}$$

where $\boldsymbol{\omega}$ is the angular rotation vector. (I draw $\boldsymbol{\omega}$ and $\hat{\mathbf{e}}_i$ in the plane of the board with angle ϕ between them; since the head of $\hat{\mathbf{e}}_i$ rotates at a distance $\sin\phi$ from the axis of $\boldsymbol{\omega}$, we have $|\dot{\hat{\mathbf{e}}}_i|=|\omega \sin \phi|$.)

Having established Eq. (1), we write $\boldsymbol{\rho}$, $\mathbf{r}$, and $\mathbf{R}$ in terms of a mixed set of unit vectors,

$$\boldsymbol{\rho}=\sum_j \rho_j\hat{\mathbf{x}}_j$$

$$\mathbf{R}=\sum_j R_j\hat{\mathbf{x}}_j$$

$$\mathbf{r}=\sum_j r_j\hat{\mathbf{e}}_j \tag{2}$$

so that

$$\boldsymbol{\rho}=\sum_j \rho_j\hat{\mathbf{x}}_j=\sum_j (R_j\hat{\mathbf{x}}_j+r_j\hat{\mathbf{e}}_j) \tag{3}$$

and upon taking the second derivative,

$$\ddot{\boldsymbol{\rho}}=\sum_j (\ddot{R}_j\hat{\mathbf{x}}_j+\ddot{r}_j\hat{\mathbf{e}}_j+2\dot{r}_j\dot{\hat{\mathbf{e}}}_j+r_j\ddot{\hat{\mathbf{e}}}_j). \tag{4}$$

The third and fourth terms are easily rewritten using Eq. (1):

$$\sum_j 2\dot{r}_j\dot{\hat{\mathbf{e}}}_j=\sum_j 2\dot{r}_j(\boldsymbol{\omega}\times\hat{\mathbf{e}}_i)=\sum_j 2(\boldsymbol{\omega}\times\dot{r}_j\hat{\mathbf{e}}_j)$$

$$\sum_j r_j\ddot{\hat{\mathbf{e}}}_j=\sum_j r_j\frac{d}{dt}(\boldsymbol{\omega}\times\hat{\mathbf{e}}_j)$$

$$=\sum_j (\dot{\boldsymbol{\omega}}\times r_j\hat{\mathbf{e}}_j+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times r_j\hat{\mathbf{e}}_j)) \tag{5}$$

which upon substitution into Eq. (4) yields the familiar result,

$$\ddot{\boldsymbol{\rho}}=\sum_j \ddot{\rho}_j\hat{\mathbf{x}}_j=\sum_j (\ddot{R}_j\hat{\mathbf{x}}_j+\ddot{r}_j\hat{\mathbf{e}}_j+2(\boldsymbol{\omega}\times\dot{r}_j\hat{\mathbf{e}}_j)+\dot{\boldsymbol{\omega}}\times r_j\hat{\mathbf{e}}_j$$
$$+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times r_j\hat{e}_j)) . \tag{6}$$

This derivation is short and does not require the introduction of two different kinds of time derivatives. It also avoids confusions that can arise from the conventional derivation, which typically leads to a formula such as

$$\ddot{\boldsymbol{\rho}}=\ddot{\mathbf{R}}+\mathbf{a}'+2\boldsymbol{\omega}\times\mathbf{v}'+\dot{\boldsymbol{\omega}}\times\mathbf{r}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{7}$$

where the primes remind us to take the derivatives $\mathbf{a}'$ and $\mathbf{v}'$ in the rotating frame.

One confusion occurs when considering the special case that both $\mathbf{R}$ and $\boldsymbol{\omega}$ vanish. In this case, $\boldsymbol{\rho}$ and $\mathbf{r}$ should be related by a simple rotation between fixed coordinate systems. In other words, $\boldsymbol{\rho}$ and $\mathbf{r}$ should represent the same direction, but have different components. This is not explicitly obvious in Eq. (7), which can lead to the erroneous conclusion that the "fixed" and "rotating" coordinate systems must be instantaneously aligned. However, with my preferred notation, Eq. (6), different unit vectors are used in expressing $\boldsymbol{\rho}$ and $\mathbf{r}$, so that it is obvious that coordinates of $\boldsymbol{\rho}$ and $\mathbf{r}$ are related by a simple rotation.

I admit that this confusion and a related one concerning why $\boldsymbol{\omega}$ is not zero in the rotating frame do not cause noticeable difficulties with students. This is because the fundamental concept of the Coriolis force is not difficult to grasp, and because one or two simple examples make it clear how to apply the formula. Students do not complain when they understand the fundamental principle, can do the homework, but cannot follow the derivation, Nevertheless, there is no reason for making a derivation more difficult than it really is.

[a)]Work done at Department of Physics and Astronomy, University of Greensboro, North Carolina 27412-5001.

[1]D. Davis, "Classical Mechanics" (Harcourt Brace: Orlando, Florida, 1986), pp. 156–159. Apparently this text has the approach which is the closest to mine. Yet even Davis introduces the two derivatives ($d\mathbf{B}/dt$ and $d'\mathbf{B}/dt$) half-way through the derivation.