Why Bach sounds funny on the piano

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Do these “harmonics” make music?

400/200 = 2/1

600/400 = 3/2

4/3

5/3

200 cps*

400 cps

600 cps

800 cps

1000 cps

cps = cycles per second = Hz

(approximate numbers for viola, rounded for convenience)
Two ways to get from G to E

Up one Sixth from G

\[ \frac{5}{3} \]

“up a Bonnie”

up two Fifths and down a Fourth

\[ \frac{3}{2} \times \frac{3}{2} \times \frac{3}{4} \]

“up two Twinkles, down an old Mac D.”
Cross-multiply the fraction

\[ \frac{5}{3} = \left( \frac{3}{2} \right) \cdot \left( \frac{3}{2} \right) \cdot \left( \frac{3}{4} \right) \]

then \[ 80 = 81 \]

Paradoxes such as this create the need for “tempered” scales that find “compromise” pitches. Three important scales are:

**Pythagorean:** (uses a circle of 3/2 musical “fifth” ratios)

**12-tone equal tempered “scientific”:** (logarithms and exponents)

**Bach’s “well tempered”:** a useful compromise for sophisticated music
Rational Fractions associated close to unity are associated with musical “commas”

\[ \log(1) = 0 \implies \log(\text{almost one}) = \text{almost zero} \]

Syntonic Comma: \[ 12 \log_2 \frac{81}{80} = 0.2151 \approx 22\% \text{ (of a semitone)} \]

Phythaogrean Comma: \[ 12 \log_2 \frac{3^{12}}{2^{19}} = 0.2346 \approx 23\% \text{ (of a semitone)} \]
Other commas

<table>
<thead>
<tr>
<th>Type</th>
<th>Percentage</th>
<th>Ratio</th>
<th>Simplified</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small undecimal comma</td>
<td>17.58%</td>
<td>99:98</td>
<td>$3^2 \times 11 : 2^3 \times 7^2$</td>
</tr>
<tr>
<td>Diaschisma</td>
<td>19.55%</td>
<td>2048:2025</td>
<td>$2^{11} : 3^4 \times 5^2$</td>
</tr>
<tr>
<td>Syntonic comma</td>
<td>21.51%</td>
<td>81:80</td>
<td>$3^4 : 2^4 \times 5$</td>
</tr>
<tr>
<td>Pythagorean comma</td>
<td>23.46%</td>
<td>531441:524288</td>
<td>$3^{12} : 2^{19}$</td>
</tr>
<tr>
<td>Septimal comma</td>
<td>27.26%</td>
<td>64:63</td>
<td>$2^6 : 3^2 \times 7$</td>
</tr>
</tbody>
</table>

**Septimal sixth-tone or jubilisma, Erlich's decatonic comma or tritonic diesis**

|                     | 34.98%     | 50:49                  | $2 \times 5^2 : 7^2$  |

\[
\frac{7}{5} \approx \sqrt{2} = \frac{2}{\sqrt{2}} \approx \frac{10}{7}
\]

THIS HAS A HARSH SOUND

Two eighteenth-century composers (Georg Philipp Telemann and Johann Joseph Fux) wrote that this interval was once associated with the devil, but Wikipedia claims no known citations from the Middle Ages.
Why is the tritone a “diabolus en musica”?

<table>
<thead>
<tr>
<th>interval</th>
<th>ratio</th>
<th>half-tones</th>
<th>error</th>
<th>How a piano plays it</th>
</tr>
</thead>
<tbody>
<tr>
<td>octave</td>
<td>2/1</td>
<td>12</td>
<td>zero</td>
<td></td>
</tr>
<tr>
<td>fifth</td>
<td>3/2</td>
<td>7.02 = 7 + 0.02</td>
<td>+2%</td>
<td>3/2 ≈ 2(^{7/12})</td>
</tr>
<tr>
<td>fourth</td>
<td>4/3</td>
<td>4.98 = 5 − 0.02</td>
<td>−2%</td>
<td>4/3 ≈ 2(^{5/12})</td>
</tr>
<tr>
<td>major sixth</td>
<td>5/3</td>
<td>8.84 = 9 − 0.16</td>
<td>−16%</td>
<td>5/3 ≈ 2(^{9/12})</td>
</tr>
<tr>
<td>major third</td>
<td>5/4</td>
<td>3.86 = 4 − 0.14</td>
<td>−14%</td>
<td>5/4 ≈ 2(^{4/12})</td>
</tr>
<tr>
<td>minor third</td>
<td>6/5</td>
<td>3.16 = 3 + 0.16</td>
<td>+16%</td>
<td>6/5 ≈ 2(^{3/12})</td>
</tr>
<tr>
<td><strong>tritone</strong></td>
<td>7/5</td>
<td>5.825 = 6 − .175</td>
<td>−17.5%</td>
<td>7/5 ≈ 2(^{6/12})</td>
</tr>
<tr>
<td>minor sixth</td>
<td>8/5</td>
<td>8.14 = 8 + 0.14</td>
<td>+14%</td>
<td>8/5 ≈ 2(^{8/12})</td>
</tr>
</tbody>
</table>

Is the “devil” in the **SEVEN** or in the **SQUARE ROOT OF TWO**?

\[2^{6/12} = 2^{1/2}.\] But since, \(\sqrt{2} = \frac{2}{\sqrt{2}}\), these “devils” always come in pairs!

\[\frac{7}{5} \approx \frac{45}{32} \approx \sqrt{2} \approx \frac{62}{45} \approx \frac{10}{7}\]

Perhaps the sound is dissonant because if one is in tune, the other is not. There might be a way to verify this.
This chart allows almost every note on the to be tested.

Grey and white circles denote the 12-tone equal tempered scale in which all frequency ratios equal

$$2^{n/12}$$

Colored circles and triangles construct the notes using “just” intervals in which the frequency ratios are

$$\frac{p}{q}$$

n, p, and q are integers.

Pythagoras is credited with discovering that $$2^{1/2}$$ cannot be expressed as an integral fraction.
George Plimpton was a journalist who enhanced his stories by actually playing quarterback for the Detroit Lions, boxing against Archie Moore, and blocking 100 MPH hockey pucks for the Boston Bruins. Plimpton did a TV special back in the 1960 and was to play a series of “Pings” on the triangle at a performance by the New York Philharmonic. During rehearsal, Leonard Bernstein stopped the orchestra....

Bernstein: George would you play that note for us again?

Plimpton: Ping!

Bernstein: One more time, please.

Plimpton: Ping!

Bernstein (cupping his hand behind his ear): Once more...

Plimpton: Ping!

Bernstein (impatient and dissatisfied): Now, which of those four pings do you mean? They're all different.

(I got this conversation off the web, my recollection is that Bernstein instructed Plimpton to choose a way to hit the triangle and hit it that way every time.)