Physics Equations May 13, 2013.

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Contents

1 Math and Vectors

1.1 Quadratic equation

The two solutions to (1a) are (1b):

$$ax^2 + bx + c = 0 \tag{1a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1b}$$

1.2 Trigonometry and Functions

The radian, sine, cosine, and tangent functions are defined by figure 1. If s is arclength and θ is the angle in radians,

$$\theta = s/r, \tag{2a}$$

$$2\pi \text{ rad} = 360 \text{ deg} = 1 \text{ rev}, \tag{2b}$$

where (rad, rad, rev) denote radian, degree, and revolution, respectively. From (2b) we conclude that the circumfrence of a circle is $2\pi r$. Integration with respect to r establishes the area of a circle as πr^2. (Similarly, one can integrate the surface area of a sphere, $A = 4\pi r^2$ to obtain the volume of a sphere as $\frac{4}{3}\pi r^3$.)

[1] Material licensed by Users on en.wikipedia.org under creativecommons.org/by-sa/3.0: Fig.3/Acdx; Fig.14/Epzcaw; Sec.9.5/Carnot_theorem_(thermodynamics)/Netheril96;

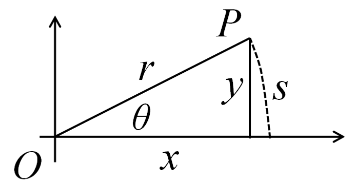

Figure 1: Radian, Sine, Cosine, Tangent

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{3a}$$

$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x. \tag{3b}$$

If the range and domain of these inverse pairs are properly defined,

$$\begin{aligned} \tan^{-1}(\tan\theta) = \theta, &\qquad \tan\left(\tan^{-1}\beta\right) = \beta, \\ \ln e^x = x, &\qquad e^{\ln y} = y. \end{aligned} \tag{4}$$

The exponential and logarithmic functions obey:

$$\log x + \log y = \log xy, \qquad n\log x = \log x^n, \tag{5a}$$

$$a^m a^n = a^{m+n}, \qquad (a^n)^m = a^{mn}. \tag{5b}$$

1.3 Calculus

It can be shown that:

$$\begin{array}{c|c|c} f(t) & \mathrm{d}f/\mathrm{d}t & \int f(t)dt \\ t^n & nt^{n-1} & (n+1)^{-1}t^{n+1} \\ e^{\alpha t} & \alpha e^{\alpha t} & \alpha^{-1}e^{\alpha t} \\ \sin\omega t & \omega\cos\omega t & -\omega^{-1}\cos\omega t \\ \cos\omega t & -\omega\sin\omega t & \omega^{-1}\sin\omega t \\ 1/t & -t^{-2} & \ln t \end{array} \tag{6}$$

where α, ω, and n are constants. Other derivatives can be performed using the product and chain rules, and the linearity of differential operators:

$$\frac{\mathrm{d}}{\mathrm{d}t}(fg) = f\frac{\mathrm{d}g}{\mathrm{d}t} + \frac{\mathrm{d}f}{\mathrm{d}t}g, \tag{7a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}f\left(g(t)\right) = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}t}, \tag{7b}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(Af(t) + Bg(t)\right) = A\frac{\mathrm{d}^n f}{\mathrm{d}t^n} + B\frac{\mathrm{d}^n g}{\mathrm{d}t^n}. \tag{7c}$$

1.4 The difference symbol: $\boldsymbol{\Delta}, \mathbf{d}, \partial, \delta$

The *Greek* Δ is one of may variants of the letter "d" that represents **difference**:

$$\Delta X = X_2 - X_1. \tag{8}$$

This difference, while not always small, is often taken as small. For example, if Δt is sufficiently small, then to good approximation:

$$\frac{f(t+\Delta t)-f(t)}{\Delta t} \equiv \frac{\Delta f}{\Delta t} \approx \frac{\mathrm{d}f}{\mathrm{d}t}, \tag{9a}$$

$$\sum f(t)\Delta t \approx \int f(t)\mathrm{d}t, \tag{9b}$$

where, $\Delta f = f(t+\Delta t) - f(t)$. There are two versions of the **Fundamental Theorem of Calculus**:

$$\int_a^b \frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t = f(b) - f(a), \tag{10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t g(\bar{t})\,\mathrm{d}\bar{t} = g(t). \tag{10b}$$

1.5 Taylor Expansions

Power series expansions for functions of one or two variables:

$$f(X) = f(0) + f^{'}(0)X + \frac{f^{''}(0)}{2!}X^2 + \cdots \tag{11a}$$

$$\Delta f(x,y) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \cdots \tag{11b}$$

In (11a) the substition $X = x - x_0$ permits an expasion around the point x_0. At $X = 0$, all the derivatives of the LHS (Left-Hand-Side) of (11a) match all those of the RHS. In (11b) we express $f(x,y) - f(x_0,y_0)$ as Δf, et cetera. The next three terms in (11b) are $\frac{1}{2}(\partial^2 f/\partial x^2)\Delta x^2 + (\partial^2 f/\partial x\partial y)\Delta x\Delta y + \frac{1}{2}(\partial^2 f/\partial y^2)\Delta y^2$

Some important Taylor expansions are:

$$(1+x)^p = x + px + \frac{p(p-1)}{2}x^2 + \cdots \tag{12a}$$

$$\sin(x) = x - \frac{x^3}{3\cdot 2} + \frac{x^5}{5\cdot 5\cdot 4\cdot 2} - \cdots \tag{12b}$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4\cdot 3\cdot 2} - \cdots \tag{12c}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3\cdot 2} + \cdots \tag{12d}$$

Though not manditory for a first reading of this text, **complex numbers** greatly simplify some physics calculations. A term by term comparison of (12b), (12c) and (12d) establishes Euler's equation, $e^{i\theta} = cos(\theta + i\sin\theta$, where $i = \sqrt{-1}$. Any complex number can be written as, $z = x + iy$, where x and y are real. Defining the complex conjugate, $z^* = x - iy$, and magnitude, $A = \sqrt{zz^*}$, we have $z = Ae^{i\theta}$ where $\tan\theta = y/x$.

1.6 Probability

Let $Pr(A)$ and $Pr(B)$ denote probabilities for outcomes A and B, respectively. If the two events are **statistically independent**, then:

$$Pr(A \text{ or } B) = Pr(A) + Pr(B) \tag{13a}$$

$$Pr(A \text{ and } B) = Pr(A)\cdot Pr(B) \tag{13b}$$

1.7 Vectors

Figure 1 depicts a displacement vector (from O to P), $\vec{r} = x\hat{x} + y\hat{y}$, though far more often, a subscript notation is used to denote the components: $\vec{A} = A_x\hat{x} + A_y\hat{y}$. We shall denote the **unit vectors** as, $\{\hat{x}, \hat{y}, \hat{z}\}$.

Equation (3a) can be used to find the x and y components if the direction and magnitude are known, e.g. $A_x = A\cos\theta$, while (3b) solves the inverse problem, e.g., $\theta = \tan^{-1} A_y/A_x$. As mentioned at (4), care must be taken with the inverse tangent function, for example by adding π to the arctangent if both A_x and A_y are negative.

Let $\vec{A}$ and $\vec{B}$ be any two vectors, and let α and β be two scalars (i.e. ordinary numbers). **Scalar multiplication** and **vector addition** proceeds as follows:

$$\text{If} \quad \vec{C} = \alpha\vec{A} \pm \beta\vec{B}, \tag{14a}$$

$$\text{then} \quad C_x = \alpha A_x \pm \beta B_x, \tag{14b}$$

$$\text{and} \quad C_y = \alpha A_y \pm \beta B_y. \tag{14c}$$

(14d)

Figure 2 gives a geometrical interpretation of vector addition, where $\vec{A} + \vec{B} = \vec{C}$. Vectors add in a **tail-on-head** fashion. Vectors subtract in a **tail-on-tail** fashion: The tails of $\vec{C}$ and $\vec{A}$ are placed together, and $\vec{B} = \vec{C} - \vec{A}$ spans the heads (arrows) of $\vec{C}$ and $\vec{A}$.

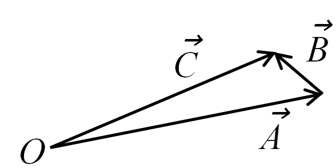

Figure 2: Geometric interpretation of $\vec{A} + \vec{B} = \vec{C}$ (vector addition) or $\vec{B} = \vec{C} - \vec{A}$ (vector subtraction)

The **magnitude** of a vector $\vec{A}$ is a non-negative scalar, and can be expressed either as either $|\vec{A}|$ or the letter A without vector symbol:

$$A = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{15a}$$
$$= \sqrt{\vec{A}\cdot\vec{A}}, \tag{15b}$$
$$\text{where}\quad \vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z \tag{15c}$$
$$= AB\cos\theta. \tag{15d}$$

Equation (15c) defines the **dot** or **inner** product. Equation (15d) allows one to find the angle θ between vectors $\vec{A}$ and $\vec{B}$.

1.8 Unit vectors and components

A unit vector is any vector with magnitude equal to 1. Any vector divided by itself is a unit vector, so that $\hat{b} = \vec{B}/B$ is a unit vector in the direction of $\vec{B}$. The dot product of a vector with a unit vector is called a **component** of that vector in the direction of the unit vector. For example, $\vec{A}\cdot\hat{b}$ is the component of $\vec{A}$ in the direction of $\vec{B}$. Thus, $A_x = \vec{A}\cdot\hat{x}$, is the component of $\vec{A}$ along the x-axis.

1.9 Cross product

The cross product of two vectors is:

$$\begin{aligned}\vec{A}\times\vec{B} = &(A_yB_z - A_yB_z)\hat{x}\\ &+(A_zB_x - A_xB_z)\hat{y}\\ &+(A_xB_y - A_yB_x)\hat{z}\end{aligned} \tag{16a}$$
$$|\vec{A}\times\vec{B}| = AB\sin\theta. \tag{16b}$$

$\vec{A}\times\vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$, and in the direction given by the **right-hand-rule** shown in Figure 3. Note that $\vec{A}\times\vec{A} = 0$ since, θ, the angle between the two vectors, equals zero. The + and − signs in (16a) are easy to remember since xyz, yzx, and zxy are all positive **cyclic permutations**.

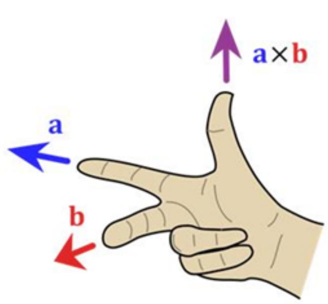

Figure 3: The right hand rule and the cross product

Always use a right-handed coordinate system in which $\hat{x}\times\hat{y} = \hat{z}$.

1.10 Polar coordinates and $\mathrm{d}\vec{\ell}$ as a change in $\vec{r}$

Since vectors may be subtracted, we may form differentials, or the differences between two nearly equal vectors. Figure 4 can be used to derive:

$$\vec{r} = x\hat{x} + y\hat{y} = r\hat{r} \tag{17a}$$
$$\mathrm{d}\vec{\ell} = \mathrm{d}\vec{r} = \hat{x}\,\mathrm{d}x + \hat{y}\,\mathrm{d}y = \hat{r}\,\mathrm{d}r + \hat{\theta}\,r\mathrm{d}\theta, \tag{17b}$$

wherethe differential change in displacement vector, $\vec{r}_2 - \vec{r}_1$, is written as $\mathrm{d}\vec{\ell}$ to avoid confusion between whether $\mathrm{d}r$ denotes $|\mathrm{d}\vec{r}|$ or $\mathrm{d}|\vec{r}|$.

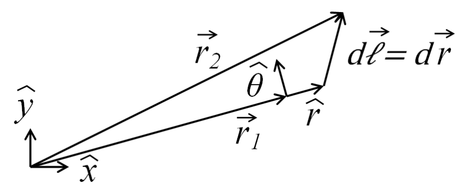

Figure 4: Unit vectors and polar coordinates

1.11 Relative velocity

Vector addition and subtraction can be used to compare how observers moving at different velocities will perceive the velocity of an object. Refer to Figure 2 and suppose two observers, "A" and "C", are initially standing on earth at point O. Observer "A" makes the displacement to the head of vector $\vec{A}$, while "B" moves to the head of $\vec{B}$, both trips requiring the same time Δt. The velocity of "A" with respect to the earth is $\vec{v}_{\mathrm{AE}} = \vec{A}/\Delta t$, while the velocity of "C" with respect to earth is $\vec{v}_{\mathrm{CE}} = \vec{C}/\Delta t$. Both observers look at vector $\vec{B}$ and conclude the the velocity of "C" relative to "A" is,

$$\vec{V}_{\mathrm{CA}} = \vec{V}_{\mathrm{CE}} - \vec{V}_{\mathrm{AE}} \tag{18}$$

This rule is only valid for speeds much less than the speed of light.

2 Acceleration and Force

2.1 Uniform acceleration

For any motion, one may define the **average velocity** as $\vec{v}_{\mathrm{ave}}$ and **average acceleration** as $\vec{a}_{\mathrm{ave}}$:

$$\vec{v}_{\mathrm{ave}} = \frac{\Delta\vec{r}}{\Delta t} \qquad \vec{a}_{\mathrm{ave}} = \frac{\Delta\vec{v}}{\Delta t} \tag{19}$$

where Δt need not be small.

For uniformly accelerated motion in the x-direction:

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \tag{20a}$$
$$v = v_0 + at \tag{20b}$$
$$v^2 = v_0^2 + 2a(x - x_0) \tag{20c}$$
$$x - x_0 = \frac{v_0 + v}{2} t \tag{20d}$$

(20b) can be derived from (20a) by taking the time derivative and defining velocity as the derivative of position. It can be shown that: (20c-20d) follow from the previous two after a bit of algebra, and (20d) establishes that for uniform acceleration, $v_{\text{ave}} = (v_0 + v)/2$, where v_0 and v are the intitial and final velocities, respectively. Equation (20b) is of course consistent with (19) since $a = a_{\text{ave}}$. Uniformly accelerated motion in two or more dimensions satisfies (20) for each spatial dimension, with the substitutions that utilize subscripts to indicate the various dimensions $x \to (x, y)$, $a \to (a_x, a_y)$ and $v_0 \to (v_{x0}, v_{y0})$.

2.2 Free Fall

If gravity is the only force acting on an object near Earth's surface, the the uniform acceleration $\vec{a}$ can be set to $(a_x, a_y) = (0, -g)$, and we have:

$$x = x_0 + v_{x0} t \qquad v_x = v_{x0} \tag{21a}$$
$$y = y_0 + v_{y0} t - \frac{1}{2} g t^2 \tag{21b}$$
$$v_y = v_{y0} - gt \tag{21c}$$
$$v^2 = v_0^2 - 2g(y - y_0) \tag{21d}$$

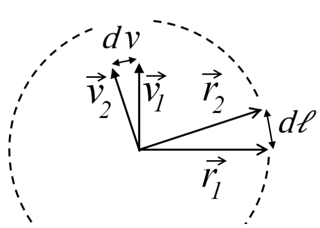

Figure 5: Uniform circularmotion

2.3 Uniform circular motion

Figure 5 shows position and velocity vectors for at two positions along the orbit (dotted circle). Define $\Delta r = |\vec{r}_2 - \vec{r}_1|$ and $\Delta v = |\vec{v}_2 - \vec{v}_1|$, and note by the discussion at Figure 2 that two triangles are formed: $\vec{r}_2 - \vec{r}_1 \equiv \Delta\ell$ and $\vec{v}_2 - \vec{v}_1 \equiv \Delta v$. Since the two triangles are similar, $\Delta\ell/r = \Delta v/v$. Note that the magnitude of the average acceleration is $a = \Delta v/\Delta t$ and that $v = \Delta\ell/\Delta t$ if the angle of rotation is sufficiently small. This leads to:

$$a = \frac{v^2}{r}. \tag{22}$$

The direction of this **centripetal acceleration** is such that $\vec{a}$ points to the center of the circle.

2.4 Newton's Laws

Newton's **second law** is,

$$m\vec{a} = \vec{F}_{\text{net}} = \sum \vec{F}_j \tag{23}$$

where m is mass, $\vec{a} = d^2\vec{r}/dt^2$ is acceleration, and $\vec{F}_{\text{net}}$ is the sum of all forces *acting on* the particle. This implies that forces influence particle motion by determining the second derivative. Equation (23) is valid only if the particle be observed in an **inertial reference frame**, that is by an observer who is moving at constant (unaccelerated) velocity.

That force was postulated proportional to the second derivative of position was likely based on observations of objects in free fall. The direct proportionality of force to mass is intuitively obvious, as illustrated by Figure 6. Equation (23) leads directly toNewton's **first** law, which is that $\vec{a} = 0$ if $\vec{F}_{net} = 0$.

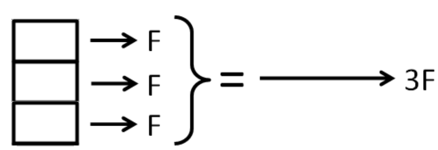

Figure 6: F is proportional to m at constant a. Tripling the mass would require three times a much force.

A plausibility argument for Newton's third law results from applying (23) to a pair of objects connected by a "massless" string or rod, as shown in Figure 7. A "massless" string has such low mass that negligible force is required to accelerate it – it acts solely as an *agent* of the interaction between objects A and B with masses m_{A} and m_{B}, respectively.

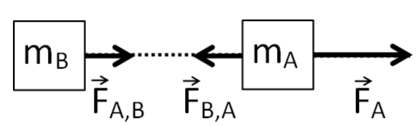

Figure 7: Newton's Third Law: Connecting string shown as dotted line

This interaction consists of a force, $\vec{F}_{\text{B,A}}$ that acts on B (but *caused* by A), and also the force, $\vec{F}_{\text{A,B}}$ that acts on A, but is *caused* by B. Also acting on A is an *external* force $\vec{F}_{\rm A}$, so that the net force on A is $\vec{F}_{\rm A} + \vec{F}_{\rm A,B}$. Application of (23) to the individual objects yields, $\vec{F}_{\rm B,A} = m_{\rm B}a$, and $\vec{F}_{\rm A} + \vec{F}_{\rm A,B} = m_{\rm A}a$. We also demand that (23) applies to the combination of the two masses (taken as a single mass, so that, $\vec{F}_A = (m_{\rm A} + m_{\rm B})a$. From this Newton's **third** law can be derived:

$$\vec{F}_{\rm A,B} = -\vec{F}_{\rm B,A} \tag{24}$$

Technically, we have **begged the question** by assuming that the internal forces ($\vec{F}_{\rm A,B}$ and $\vec{F}_{\rm B,A}$) not be included when (23) was applied to the combined mass ($m_{\rm A} + m_{\rm B}$). While this renders the "proof" logically flawed, consider the alternative: If internal forces governed the motion of macroscopic objects, objects would likely move according to their own internal forces in an unfathomable way, and "physics" would not exist.

2.5 The forces of introductory physics

We have already encountered **tension** in a a massless string, and argued that the force exerted by this tension is equal at each end of the string. The arguement that this is a consequence of Newton's third law is dubious, since a string wrapped around a frictionless (and massless) pulley also exhibits this effect. It is the (nearly) zero mass of the string that prohibits any significant difference in the forces at each end. Otherwise the string would accelerate with (nearly) infinite acceleration. At earth's surface, the acceleration of an object in free fall is $a = g \approx 9.8\text{m/s}^2$. Substitution into (23) yields an important formula for **weight**, which is the force of gravity, F_g:

$$\vec{F}_{\rm g} = m\vec{g}, \tag{25}$$

where $\vec{g} = -g\hat{y}$ if the y-direction represents "up".

Another essential force is the **contact force** between two objects. This force is shown acting on the rectangular object in Figure 8, where it is assumed that the contact is between two flat surfaces. The direction perpendicular to this surface is called the **normal** to the surface. Let, $\hat{n}$, denote the **unit normal**, which is unit vector normal to the surface. The component of the contact force parallel (or antiparallel) to is called the **normal force**, which can be written as $\vec{F}_{\rm n}$. By the discussion after (15d), $\vec{F}_n = \vec{F}_{\rm contact} \cdot \hat{n}$. The "remainder" of the contact force, i.e. ($\vec{F}_{\rm contact} - \vec{F}_n$) is the force of **friction**, which we denote as $\vec{f}$. Both the friction force and the normal force are shown in Figure 8.

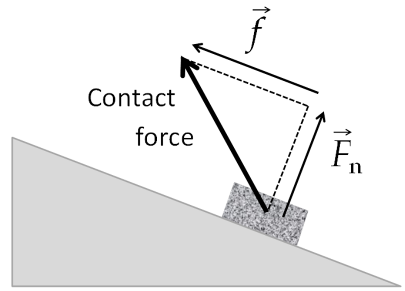

Figure 8: Friction $\vec{f}$ and normal force $\vec{F}_{\rm n}$

2.6 Friction

We now present a simple model applies only to friction between solid surfaces, and is often invalid for objects moving through fluids. If the object is not moving, the force of friction is called **static friction**, which we denote as $f_{\rm s}$. When friction is sufficient to prevent an object from moving on a stationary surface, $f_{\rm s}$ is exactly equal and opposite to the all other forces that act parallel to the surface, but the object will begin to move ("slip") if these forces exceed a certain critical value. Hence we have an inequality for static friction:

$$f_{\rm s} \le \mu_{\rm s} F_{\rm n} \tag{26}$$

where $F_{\rm n}$ is the normal force defined above, and $\mu_{\rm s}$ is a constant that depends only on the two substances in contact (*e.g. glass on aluminum, etc.*) Surprisingly, $\mu_{\rm s}$ does not depend on the area of contact, or its shape.

If the object is in motion, the force of **kinetic friction** is opposes the motion and is given by,

$$f_{\rm k} = \mu_{\rm k} F_{\rm n} \tag{27}$$

Surprisingly, $\mu_{\rm k}$ does not depend on speed, although both (26) and (27) are generally just approximations.

3 Work and Energy

Equation (20c) describes the evolution of v^2 under uniform acceleration by predicting a final velocity, $v_{\rm f} = v$, given a known initial velocity, $v_{\rm i}$, where the subscripts "f" and "i" denote final and initial, respectively. Multiply by $m/2$ and use (23) to obtain $\frac{1}{2}mv_{\rm f}^2 = \frac{1}{2}mv_{\rm i}^2 + \sum F\Delta x$, where $\Delta x = x_{\rm f} - x_{\rm i}$. It can be shown that the generalization of this result for nonuniform force in three dimensions is,

$$\frac{1}{2}mv^2_{\rm x,\ final} = \frac{1}{2}mv^2_{\rm x,initial} + \sum F_x\,\Delta x \tag{28a}$$

$$\frac{1}{2}mv^2_{\rm final} = \frac{1}{2}mv^2_{\rm initial} + \sum \vec{F}\cdot\Delta\vec{\ell} \tag{28b}$$

where (28b) utilizes the dot product defined at (15c). The sum in the RHS of (28b) can be written in a number of forms:

$$\sum \vec{F}\cdot\Delta\vec{\ell} = \sum F_x\Delta x + \sum F_y \Delta y + \ldots \tag{29a}$$

$$\rightarrow \int \vec{F}\cdot \mathrm{d}\vec{\ell} \rightarrow F_{\|}\Delta\ell \tag{29b}$$

In the final and most simple form of (29b), the force is constant, and $F_{\|}$ is the component of $\vec{F}$ along the direction defined by $\Delta\ell$ as discussed at 1.8. The student is should learn the calculus well enought that the alternative forms of (29) come naturally.

Equation (28) is known as the **Work Energy Theorem**. The **kinetic energy**, K, and the **work**, W, done by a force are defined by:

$$\text{Kinetic Energy} = K = \frac{1}{2}mv^2, \tag{30a}$$

$$\text{Work} = W = \int \vec{F}\cdot \mathrm{d}\vec{\ell}. \tag{30b}$$

3.1 Conservative forces

A force is conservative if it can be expressed as the **gradient** of a scalar field. A **scalar field** is a function of the four variables $(x,y,z,t) = (\vec{r},t)$, although the function, $U = U(\vec{r},t)$ need not be a function of all four variables. By convention we introduce a minus sign in the definition:

$$\vec{F} = -\vec{\nabla}U = -\left(\frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}\right), \tag{31a}$$

$$\int_{\vec{r}_0}^{\vec{r}} F\cdot \mathrm{d}\vec{\ell} = U(\vec{r}) - U(\vec{r}_0), \tag{31b}$$

$$\oint F\cdot \mathrm{d}\vec{\ell} = 0 \quad \text{for all closed paths.} \tag{31c}$$

Here, the **gradient operator** has been defined in (31a) as $\vec{\nabla}U = (\partial U/\partial x)\hat{x} + (\partial U/\partial y)\hat{y} + \cdots$. As one might guess from the fundamental theorem of calculus (10a) the two statements (31a) and (31b) are equivalent. The circle on the integral sign in (31c) signifies a **closed integral**. A line integral (and corresponding path) for which the starting and end points are the same is called **closed**. A complete understanding of (31) can be achieved using (11b) to calculate, $dU = (\partial U/\partial x)dx + (\partial U/\partial y)dy = \vec{\nabla}U\cdot \mathrm{d}\vec{\ell}$

Some important conservative forces are listed below:

$$\begin{array}{c|c|cl}
\vec{F}(\vec{r}) & \vec{F}\cdot \mathrm{d}\vec{\ell} & U(\vec{r}) & \\
-mg\hat{y} & -mg\,dy & mgy + U_0 & (a \\
-k_s x & -k_s x\,\mathrm{d}x & \frac{1}{2}k_s x^2 + U_0 & (b \\
\hat{r}/r^2 & \mathrm{d}r/r^2 & 1/r + U_0 & (c \\
\hat{r}/r & \mathrm{d}r/r & \ln(r_0/r) & (d \\
\vec{F}_0 & F_{x0}\mathrm{d}x + F_{y0}\mathrm{d}y & -\vec{F}_0\cdot\vec{r} + U_0 & (e
\end{array} \tag{32}$$

In each case, $U_0 = U(r_0)$, is a constant of integration that is usually set to zero. Only (32-a) and (32-b) are required for a first reading. They model the weight associated with earth's surface gravity, and the force associated with a stretched spring, respectively. **Hooke's Law** is $F = -k_s x$, where k_s is the **spring constant**.

Hooke's law is a valid approximation for spring that is not **overstretched**, but is often a poor approximation for rubber bands. The position (or displacement), x, in Hooke's Law is measured from a point $(x=0)$ called the **equilibrium** position.

The forces (32-c) and (32-d) will be useful for Newton's theory of gravity and for electrostatics. Force (32-e) is the generalization of (32-a) when a constant force field is not aligned with the y-axis. In each of these cases the student should be able to integrate $\int \vec{F}\cdot \mathrm{d}\vec{\ell} = \int \vec{F}\cdot \mathrm{d}\vec{r}$ and verify that it equals $-U$.

3.2 Nonconservative forces

The force of **friction** is nonconservative, meaning that none of the three equivalent statements (31) hold. There is no potential energy $U = U(\vec{r})$ that can be associated with friction. Since force of friction, $\vec{f}$, is opposite in direction to the particles motion, $\vec{f}\cdot \mathrm{d}\vec{\ell} = -f\mathrm{d}\ell$ is always negative. If the friction remains constant over a closed path, then $\oint \vec{f}\cdot \mathrm{d}\vec{\ell} = -f\oint d\ell$, where $\oint d\ell$ is the circumfrence. The minus sign tells us, alas, that the work done by friction is always negative.

3.3 Energy conservation

We generalize (28b) to include multiparticle systems by summing over all the particles and over all the forces. If a force can be obtained from a potential U, then we are at liberty to use employ (31c), otherwise we keep in the form of a line integral (30b). The kinetic energy K, defined at (30a), as well as potential energy U and work W are summed over all particles and all forces. The result is a statement of conservation of energy:

$$\sum K_\mathrm{f} + \sum U_\mathrm{f} = \sum K_\mathrm{i} + \sum U_\mathrm{i} + W^\mathrm{nc} \tag{33}$$

where the subscript "f" and "i" refer to "final" and "initial" states, respectively, K is kinetic energy and U is potential energy. The superscripts in W^{nc} refer to **non-conservative work**, i.e., the work done by the conservative forces has been incorperated into the changes potential energy.

3.4 Power

Power is the rate at which energy enters or leaves a system. For example, the power, P, associated with the work done by a force, $\vec{F}$, can be calculated using (28b) in a calculation of the power required to change the kinetic energy of a particle:

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\vec{F}\cdot \mathrm{d}\vec{\ell}}{\mathrm{d}t} = \vec{F}\cdot\vec{v} \tag{34}$$

4 Momentum and Impulse

The momentum of a single particle is defined as $\vec{p} = m\vec{v}$. Suppose we have a system of N interacting particles, where, m_{j}, is the mass of each particle ($j = 1, 2, \ldots, N$). Newton's second law (23) for the j-th particle can be written as,

$$\vec{p}_{\mathrm{j}} = m_{\mathrm{j}}\vec{v}_{\mathrm{j}} \to \vec{F}_{\mathrm{j}} = \frac{\mathrm{d}\vec{p}_{\mathrm{j}}}{\mathrm{d}t} \tag{35}$$

We now sum (35) over all N particles, and note an important fact: The **internal forces cancel** whenever all forces are summed. This is a consequence of Newton's third law (24). "Internal" forces are those that act between particles in the system. It can be shown that,

$$\vec{p}_{\mathrm{net}} = \sum_{\mathrm{j}=1}^{N} m_{\mathrm{j}}\vec{v}_{\mathrm{j}} \tag{36a}$$

$$\frac{\mathrm{d}\vec{p}_{\mathrm{net}}}{\mathrm{d}t} = \sum_{\mathrm{j}=1}^{N} \vec{F}^{\,\mathrm{j}}_{\mathrm{external}} \tag{36b}$$

where $\vec{F}^{\,\mathrm{j}}_{\mathrm{external}}$ is net external force acting on the j-th particle. If we define the **center-of-mass** position vector, $\vec{r}_{\mathrm{cm}}$ as a **weighted average**, it can furthermore be shown that :

$$\vec{r}_{\mathrm{cm}} = \frac{\sum m_{\mathrm{j}}\vec{r}_{\mathrm{j}}}{\sum m_{\mathrm{j}}} = \frac{\sum m_{\mathrm{j}}\vec{r}_{\mathrm{j}}}{M}, \tag{37a}$$

$$\vec{v}_{\mathrm{cm}} = \frac{\sum m_{\mathrm{j}}\vec{v}_{\mathrm{j}}}{M} = \frac{\mathrm{d}\vec{r}_{\mathrm{cm}}}{\mathrm{d}t}, \tag{37b}$$

$$\vec{a}_{\mathrm{cm}} = \frac{\mathrm{d}\vec{v}_{cm}}{\mathrm{d}t} = \frac{1}{M}\sum \vec{F}^{\,\mathrm{j}}_{\mathrm{external}}, \tag{37c}$$

where it is convenient to define $M = \sum m_{\mathrm{j}}$ as the mass of the system. The ability to apply Newtonian mechanics to systems of particles, alluded to at (24), is now established, but only for forces that do not involve radiation.

4.1 Comparison of two conservation laws

If $\sum \vec{F}^{\,\mathrm{j}}_{\mathrm{external}} = 0$ in (37), the system is subject zero net external force, and the result is a constraint on the motion of individual particles called **conservation of linear momentum**: $\vec{p}^{\,\mathrm{net}}_{\mathrm{final}} = \vec{p}^{\,\mathrm{net}}_{\mathrm{initial}}$. A similar situation occurred at (33)) regarding **mechanical energy** (ME): If forces in a system are conservative, $W^{\mathrm{nc}} = 0$, and $\mathrm{ME}_{\mathrm{final}} = \mathrm{ME}_{\mathrm{initial}}$, where the mechanical energy is, $\mathrm{ME} = \sum K_{\mathrm{j}} + \sum U_{\mathrm{k}}$ (with different subscripts because kinetic energy K_{j} is summed over particles, while potential energy U_{k} is over interactions.)

It is possible to conserve momentum without conserving ME (mechanical energy) . If ME not conserved, the collision is **elastic**, but if ME is not conserved in a collision, then the collision is **inelastic**.

If a ball rolls down a ramp without friction, ME is conserved but there is an apparent violation of momentum conservation due to the combined forces on the ball by the ramp and earth's gravity. This can be resolved by including the momentum of the earth as part of the system.

4.2 Momentum Impulse Theorem

If a force $\vec{F}$ acts for a time Δt, where Δt is not neccessarily small, the impulse $\vec{I}_{\mathrm{imp}}$ is defined as,

$$\vec{I}_{\mathrm{imp}} = \int_0^{\Delta t} \vec{F}\mathrm{d}t \equiv \langle \vec{F} \rangle\, \Delta t = \vec{p}_{\mathrm{f}} - \vec{p}_{\mathrm{i}} \tag{38}$$

where "f" denotes "final" and "i" denotes the "initial" momentum. This equation defines $\langle \vec{F} \rangle$ as a time-weighted average force: $\langle \vec{F} \rangle = \dfrac{1}{\Delta t}\displaystyle\int \vec{F}\mathrm{d}t$. The last step in (38) was derived by applying (10) to (35).

5 Rotating Objects

We treat the motion of a solid object about fixed axis of rotation by applying Newton's laws to a collection of N particles constrained to rotate as a **rigid rotor** (with no relative motion among the particles). An external force is applied to one or more points on the object. A sketch of a rigid rotor with only $N = 3$ particles is shown in Figure 9, where external forces $\vec{F}^{\mathrm{external}}_{\mathrm{j}}$ are applied to two of the three particles ($j = 2$ and $j = 3$).

Also indicated in the figure is the component, $F^{\mathrm{external}}_{\perp,j}$, of the external force that is perpendicular to the position vector $\vec{r}_j$ of the j-th particle. At (43) below, we shall establish that it is the **torque**, $\tau = rF_\perp = FL$, that drives rotational motion in the same way that force drives linear motion. Here L is the **lever arm** and $F = |\vec{F}|$ is the magnitude of the applied force.

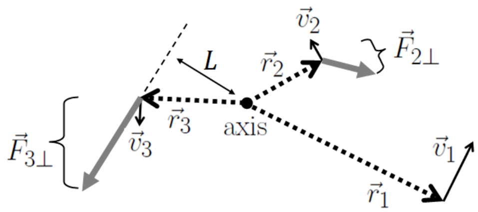

Figure 9: Rigid rotor with three atoms

Angular		**Linear**	
angle	$\theta = s/r$	position	x
angular velocity	$\omega = \mathrm{d}s/\mathrm{d}t$	velocity	$v = \mathrm{d}x/\mathrm{d}t$
angular acceleration	$\alpha = \mathrm{d}\omega/\mathrm{d}t$	acceleration	$a = \mathrm{d}v/\mathrm{d}t$
torque	$\tau = I\alpha$	force	$F = ma$
angular momentum	$L = I\omega$	momentim	$p = mv$
kinetic energy	$K = \frac{1}{2}I\omega^2$	kinetic energy	$K = \frac{1}{2}mv^2$

Table 1: Rotational Variables

5.1 The angular variables ω, T, and f.

To analyze the motion of a rigid rotor under in influence of external forces, combine (3) and (17a) to obtain an expression for the motion of the j-th particle if it is circling the axis with uniform angular speed:

$$\vec{r}_j = R_j \cos(\omega t + \delta_j)\,\hat{x} + R_j \sin(\omega t + \delta_j)\,\hat{y} \tag{39}$$

This represents motion in a circle of radius R_j, where the particle is at coordinates $(R_j \cos\delta_j, R_j \sin\delta_j)$ at time $t = 0$. Using the fact that the **frequency**, f, is the number of revolutions per second, and that the periodicity of the sine and cosine function is 2π, the following relationships can be derived:

$$\omega T = 2\pi \qquad fT = 1 \tag{40a}$$

$$v = \omega R \tag{40b}$$

where the **angular frequency**, $\omega = \mathrm{d}\theta/\mathrm{d}t$ is the time derivative of the angle, $\theta = \omega t + \delta_j$, shown in Figure 1, T is the **period**, or time to undergo one revolution. The variables R and v refer to the radius and speed (respectively) of any particle that moving in a perfect circle. Equation (40b) allows us to simplify the formula for the **kinetic energy** (30a) of the N particles associated with the rigid rotor shown in Figure 9. Summing over, j $= 1 \to N$, particles in the rigid rotor, and substituting $v_j = \omega r_j$:

$$K_{\text{net}} = \frac{1}{2}\sum m_j v_j^2 = \left\{\frac{1}{2}\sum m_j r_j^2\right\}\omega^2 \tag{41a}$$

$$= \frac{1}{2} I_{\text{moi}}\omega^2 \tag{41b}$$

$$\text{where,}\quad I_{\text{moi}} = \sum m_j r_j^2 \tag{41c}$$

(or simply I) is called the moment of inertia.

By (34) an expression for the power , $P^{\text{net}}_{\text{external}}$, associated with the external forces $\vec{F}^{\text{external}}_j$ is obtained:

$$P^{\text{net}}_{\text{external}} = \sum \vec{F}^{\text{external}}_j \cdot \vec{v}_j = \sum F^{\text{external}}_{\perp,j} v_j \tag{42a}$$

$$= \omega\left\{\sum F^{\text{external}}_{\perp,j} r_j\right\} = \omega\tau_{\text{net}} \tag{42b}$$

where we have definedthe **torque** on a particle that is a distance $r = |\vec{r}|$ from the axis as,

$$\tau = F_\perp r = FL = |\vec{r}\times\vec{F}| \tag{43}$$

The **vector torque** is $\vec{\tau} = \vec{r}\times\vec{F}$. The direction of this vector is illustrated in in Figure 10.

Figure 10: If the thumb of the right-hand points parallel to $\vec{\tau}$, the fingers point in the associated direction of rotation.

It is straightfoward to calculate the time derivative of kinetic energy in (41c) to obtain $\mathrm{d}K/\mathrm{d}t = I_{\text{moi}}\cdot\omega\cdot\mathrm{d}\omega/\mathrm{d}t$. Equate this to the power generated by the external forces, and we have a "rotational analog" to $F = ma$:

$$\tau = I_{\text{moi}}\frac{\mathrm{d}\omega}{\mathrm{d}t} = I_{\text{moi}}\alpha \tag{44}$$

where $\alpha = \mathrm{d}\omega/\mathrm{d}t = \mathrm{d}^2\theta/\mathrm{d}t^2$ is the **angular acceleration**. Other analogs are shown in Table 1. Each equation in (20) holds for rotation about a fixed axis provided the substitutions indicated in Table 1 are made. For example, $\theta = \theta_0 + \omega_0 + \frac{1}{2}\alpha t^2$.

One final issue needs to be resolved to establish that the equations Table 1 follow directly from Netwon's laws of motion. It was asserted that only external forces contribute to the net torque, $\tau_{\rm net}$ in (42b). A similar situation occured when Newton's third law (24) was used to assert that internal forces do not contribute to center-of-mass motion in (36). A cancellation not unlike the one discussed above (36) does occur, but the analysis is more elaborate. A simple physical argument is more appropriate here: If internal forces generate or consume power, it should be recognized at an atomic scale. Since the atoms are "locked" in place on the rigid rotor, there is no mechanism by which stored energy could arise from, e.g., compression. The axis about which a rotating object rotates does exert a force, but that force can absorb energy only if friction is present.

6 Oscillations and Waves

Consider a mass, m, that is subject to three forces: The first is caused by a spring, $F_s = -k_s x$, where k_s, is the **spring constant** discussed after (32). The second force, $F_d \cos(\Omega t)$, is an external driving force, where Ω is the **driving frequency**. The third force, $f = -bv = -v\mathrm{d}x/\mathrm{d}t$, is due to **friction**. (This frictional force contradicts the model for friction at (27) but makes for simpler and more useful mathematics.) From (23), we can write:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega_0^2 x + \omega_0 Q^{-1}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{F_d}{m}\cos(\Omega t) \tag{45}$$

where $\omega_0 = (k_s/m)^{1/2}$, is the **natural frequency** of the mass-spring system, and $Q^{-1} = b/(m\omega_0)$ is the **quality factor**. A system with infinite Q has no damping ($b = 0$).

Equation (45) is an **inhomogenous linear differential equation**. It can be shown that (45) is most easily solved by trying solutions of the form $x = Ce^{i\varpi t}$, and that the solution is:

$$x(t) = x_{\rm homogeneous} + x_{\rm steady\ state} = x_{\rm h} + x_{\rm ss} \tag{46}$$

where $x_{\rm h} = x_{\rm homogeneous}$, depends on two arbitrary constants (C_1 and δ_c) that might be determined from the initial conditions (e.g., position and velocity at time $t = 0$.)

$$x_h = C_1 e^{-\nu t}\cos(\omega t - \delta_c) \tag{47a}$$

$$\text{where,}\quad \omega = \omega_0\sqrt{1 - \frac{1}{4Q^2}}, \tag{47b}$$

$$\text{and,}\quad \nu = \frac{\omega_0}{2Q}. \tag{47c}$$

This is also the solution to the undriven oscillator (i.e., $F_d = 0$). Equation (47)is intended for the **underdamped** oscillator, in which $4Q^2 > 1$. In this case both ω and ν are real-valued.

If a driving term is present, we must add the "steady state" solution to (45), $x_{\rm ss} = x_{\rm steady\ state}$:

$$x_{\rm ss} = A\cos(\Omega t - \delta_{\rm A}) \tag{48a}$$

$$\text{where,}\quad A = \sqrt{\frac{F_d/m}{\left(\omega_0^2 - \Omega^2\right)^2 + \left(\Omega\omega_0 Q^{-1}\right)^2}}, \tag{48b}$$

$$\text{and,}\quad \delta_A = \tan^{-1}\left(\frac{\Omega\omega_0 Q^{-1}}{\Omega^2 - \omega_0^2}\right). \tag{48c}$$

Plots of the amplitude, A and phase shift δ_A are shown in Figures 11 and 12

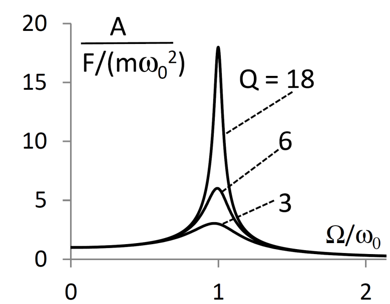

Figure 11: Amplitude (A) versus driving frequency (Ω)

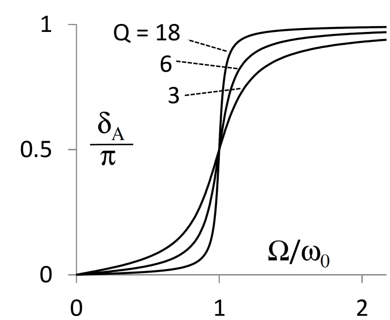

Figure 12: Phase (δ_A) versus driving frequency (Ω)

6.1 Dispersion relations and phase velocity

Here we consider waves that are **linear** in wave amplitude and are supported by a medium that is **homogeneous** in both time and spatial coordinates. For example, (51a) below, is such a wave, provided the coefficient c does not depend on x or t. A linear wave equation is one for which all terms contain ψ to the first power only, and terms like $\psi^2, \psi^3, \ldots$, are absent. The following solution to any linear wave in a homogeneous medium is called a **plane** wave because regions of constant phase, $\phi = kx - \omega t + \delta = constant$, are planes in a three-dimensional space.

$$\psi(x,t) = \mathrm{Re}\{Ae^{ikx - i\omega t + i\delta}\} \tag{49a}$$
$$= A\cos(kx - \omega t + \delta), \tag{49b}$$
$$\text{where } \ \omega = \omega(k). \tag{49c}$$

The statement, $\omega = \omega(k)$, means that angular frequency ω is a function of wavenumber k. This relationship between ω and k is called a **dispersion relation**. By analogy with (40), wavenumber, k, and wavelength, λ, are related by:

$$k\lambda = 2\pi. \tag{50}$$

A simple yet important wave equation is,

$$\partial^2\psi/\partial t^2 = c^2\partial^2\psi/\partial x^2, \tag{51a}$$
$$\omega = \pm ck, \tag{51b}$$

where this dispersion relation has two branches because substitution of (49) into (51a) yields, $\omega^2 = c^2k^2$. One unique feature of this particular wave equation is that for almost any function, $f = f(u)$, a solution to (51a) is, $\psi(x,t) = f(x \pm ct)$. If ω and k are both real, any dispersion relation has a **phase velocity**, defined as the speed of any point of constant phase:

$$v_{\text{phase}} = \frac{\omega}{k}. \tag{52}$$

The phase velocity is the velocity at which an individual "crest" remains constant, since the phase in that "reference frame" remains constant: $\phi = kx' - \omega t + \delta = x_0 + \delta$, where $x' = x_0 + (\omega/k)t$.

For waves of the that satisfy (51), $v_{\text{phase}} = c$. If (51) is used to model light waves, $c \approx 3\times 10^8$m/s. For sound waves in air, $c = \sqrt{\gamma P/\rho} \approx 340$m/s, where $\gamma \approx 1.4$, P is the pressure, and ρ is the mass density. This speed depends on temperature and molecular or atomic mass, but not density. If (51) is used to model transverse waves on a string, then $c = \sqrt{F_{\text{T}}/\mu}$ where F_{T} is the force of tension and μ is the linear mass density (mass per unit length). Not all waves are described by (51). For example, bending waves in a thin rod obey, $\omega^2 = ak^4$ (with four branches), and deep water gravity waves obey $\omega^2 = gk$, where $g \approx 9.8\text{m/s}^2$.

6.2 Beats

The **superposition principle** holds for any linear wave, and implies that if two solutions are added, the result is also a solution to the same wave equaiton. It can be shown that:

$$\psi = \cos(\omega_2 t) + \cos(\omega_1 t) = A(t)\cos\bar{\omega}t, \tag{53a}$$
$$\text{where, } \ A(t) = 2\cos\left(\frac{\Delta\omega}{2}t\right), \tag{53b}$$
$$\bar{\omega} = \frac{\omega_2 + \omega_1}{2}, \tag{53c}$$
$$\text{and, } \ \Delta\omega = \omega_2 - \omega_1. \tag{53d}$$

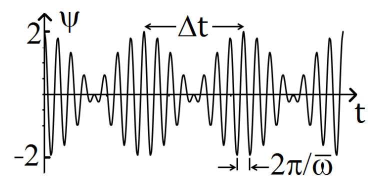

Figure 13: Beating of two waves

If $\Delta\omega << \bar{\omega}$ then $A(t)$ is a slowly varying modulus in that ψ oscillates rapidly between $\pm|A(t)|$, as shown in Figure 13. If Δt is the time between succesive zeros of $|A(t)|$, then

$$(\Delta\omega)(\Delta t) = 2\pi \tag{54}$$

The beat frequency is Δt^{-1}.

6.3 Young's two-slit diffraction

A nearly identical calculation recovers **Young's two-slit diffraction**, which is interference associated with the addition of waves with different path lengths differ. For example, if one wave (49) has a phase $\phi_1 = kx - \omega t$, while the phase of the other is $\phi_2 = k(x + \Delta x) - \omega t$, it can be shown that successive maxima occur when $k\Delta x = 2\pi$, or equivalently, $S\sin\theta = n\lambda$, where n is an integer, and θ is defined in figure 14.

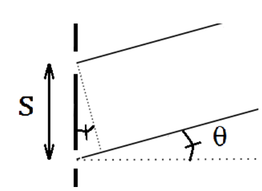

Figure 14: Two slit diffraction geometry

6.4 Group Velocity

While plane waves extend from $x = -\infty$ to $x = +\infty$, it is possible to superimpose an infinite number of plane waves to produce a wavepacket of finite size as shown in Figure 15. To simplify the discussion, we add just two waves of equal amplitude, and create an infinite train of packages not unlike that shown in Figure 13, except that now the horizontal axis represents position, so that in the figure we would make the replacements $\Delta t \to \Delta x$ and $2\pi/\bar{\omega} \to 2\pi/\bar{k} = \bar{\lambda}$, which is the average of the two (nearly equal) wavelengths. The dispersion relation (49c) is used to calculate the **group velocity**, $v_{\text{group}} = \Delta\omega/\Delta k = (\omega_2 - \omega_1)/(k_2 - k_1)$, where one wave has frequency-wavenumber (ω_1, k_1), while the other is (ω_2, k_2). It can be shown that:

$$\psi = e^{ik_2x - i\omega_2 t} + e^{ik_1x - i\omega_1 t} \tag{55a}$$

$$= 2\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)e^{i\bar{k}x - i\bar{\omega}t} \tag{55b}$$

Note that $\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$ represents a slowly varying modulus, $A(x,t)$, moves at the group velocity $\Delta\omega/\Delta k$). The appearance of ψ when graphed versus x is identical to Figure 13, with an obvious relationship between the "size" of a single beat and the spread in wavenumber. By analogy with (54): $\Delta x \Delta k = 2\pi$.

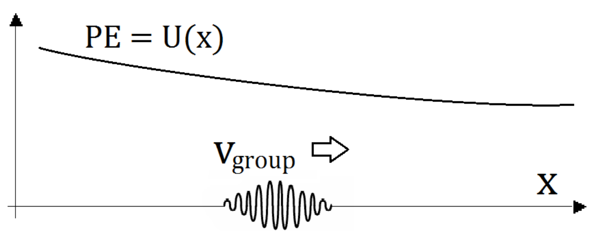

Figure 15: A wavepackt accelerating "downhill"

6.5 Can wavepackets obey F=ma?

The answer is yes, but only if certain conditions are met, and only to approximation. We shall state identical equations of motion two forms:

$$\mathrm{d}x/\mathrm{dt} = \partial H/\partial p \qquad \mathrm{d}p/\mathrm{dt} = -\partial H/\partial x \tag{56a}$$

$$\mathrm{d}x/\mathrm{dt} \approx \partial\omega/\partial k \qquad \mathrm{d}k/\mathrm{dt} \approx -\partial\omega/\partial x \tag{56b}$$

Equation (56a) is a simplified version of Hamilton's restatement of (23), which represents 146 years of refinement of Newton's *Principia*. Using (30), (31a), and (35), we write energy E as a function of momentum and position called a "hamiltonian", H: $E = H(p,x) = p^2/(2m) + U(x)$, where U is the potential energy. The reader can verify that (56a) recovers (23).

To construct a plausibility argument that (56b) describes wavepacket motion, we first argue that ω will remain approximately constant as time evolves. This is justified by the fact that a linear equation like this exhibits solutions of the form $\psi(x)e^{i\omega t}$. Use (11b) to take the differential and set it to zero: $\mathrm{d}\omega = 0 = \partial\omega/\partial k\ \mathrm{d}k + \partial\omega/\partial x\ \mathrm{d}x$. Divide by t and note that $\mathrm{d}x/\mathrm{d}t = \partial\omega/\partial k$, so that: $\mathrm{d}k/\mathrm{d}t = \partial\omega/\partial x$. Although this plausibility argument is restricted to one dimension, (56) can be extended to arbitray dimensions, e.g., with the hamiltonian, $H = H(x_1, p_1, x_2, p_2, \ldots, x_N, p_N)$, where N can be very large (or infinite).

Equation (56b) approximately describes wavepacket motion only if the wavepacket is many wavelenghts long, yet small enough so that the coefficients are essentially uniform over the length of the wavepacket. Most wavepackets lose their coherence and eventually spread out. The insight that wavepackets could behave as particles was embedded in the mathematics of classical dynamics since before 1834.

6.6 Standing Waves

Linear wave equations also support standing waves, often with some sort of **boundary condition**. For example, if a string is stretched between $x = 0$ and $x = L$, the boundary condition is that $\psi = 0$ at the ends. If the string is held motionless and suddenly released at time, $t = 0$, the general solution to (51) is $\sum A_n \cos(\omega_n t)\sin k_n x$, where the sum is from $n = 1 \to \infty$, and $\omega_n = \omega(k_n)$. To satisfy the boundary conditions, we have, $k_n L = n\pi$, which implies that $k_n = n\pi/L$. Wavefunctions for $(n = 1, 2, 3)$ are shown in Figure 16. The amplitude coeffients A_n are arbitrary constants that depend on initial conditions. They can be calculated using **Fourier Analysis**, given the initial shape of the string when it is released. For reference: $A_n = \frac{2}{L}\int \psi(x,0)\sin(n\pi x/L)\mathrm{d}x$, where the integral is from 0 to L.

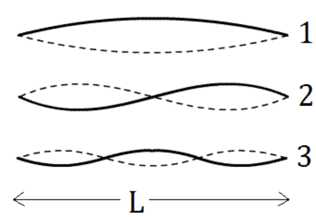

Figure 16: Three fixed-end normal modes

7 Fluids

7.1 Pressure and Bulk Modulus

Density, ρ, (also called *mass density*) is the ratio,

$$\rho = m/V \tag{57}$$

where m is mass and V is volume. Pressure, P, is defined as the ratio of force F on a surface to its area A:

$$P = F/A \tag{58}$$

The SI unit of pressure is the pascal (1 Pa =N/m^2). The the bar is defined as 100 kPa, and the atmosphere (atm) equals 101.325 kPA $\approx$ 14.7 lb/in^2.

If a substance changes by increases by a small amount ΔP, one would expect the volume to decrease by a small amount ΔV. An **intrinsic parameter** is a parameter that depends only on the intrinsic properties of the substance, and not on the amount of matter present. Since $\Delta V/V$ and ΔP are both intrinsic variables, so is the ratio, B, which is called the **bulk modulus**:

$$B = -\frac{\Delta P}{\Delta V/V} \tag{59}$$

7.2 Pressure versus Depth

Consider a cylinder of height Δh and cross sectional area A. Orient the cylinder so that the axis is vertical, and assume that the top of the cylinder is at a depth of h below the surface of the fluid. Let the pressure at the top surface be P_0, so that the downward force at the top surface is $P_0 A$. Let us also assume that the cylinder consists of the some fluid of density ρ, and that the fluid is immersed in the same fluid. We call such a construct a *fluid element*. If the fluid is at rest, the net force on the fluid element must vanish. To compensate for the force of gravity, an extra force at the botoom surface must equal the *weight* of the fluid in the cyllinder.

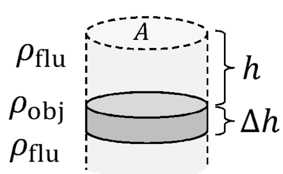

Figure 17: The pressure exerts a force equal to the weight of the fluid above the object. But the upward force of pressure exerted at the bottom is greater.

Hence, pressure increases with increasing depth according to:

$$P = P_0 + \rho g \Delta h \tag{60}$$

where P_0 is the pressure at some depth, and Δh is the change in depth. This equation assumes that the density ρ is constant. Note that pressure depends only on depth, and not on the shape of the container. This is known as **Pascal's Principle**.

7.3 Buoyancy and Archimedes' Prinicple

Archidemes principle states that:

> **A body wholly or partially submerged in a fluid is buoyed up by a force equal the the weight of the displaced fluid.**

To put this into equation form, we must distinguish between the mass density of the *fluid* and the *object* whose buoyant force we wish to calculate. We must also distinguish between the *volume* of the object, and the *submerged volume* of the object. This latter refers to that portion of the object that is below the objects "waterline", and is also called the **displaced volume**. Thus we have the following relationships between the objects mass, volume and displaced volume:

$$M_{\text{object}} = \rho_{\text{object}} V_{\text{object}} \tag{61a}$$
$$\text{if fully submerged:} \quad V_{\text{displaced}} = V_{\text{object}} \tag{61b}$$
$$\text{if partially submerged:} \quad V_{\text{displaced}} < V_{\text{object}} \tag{61c}$$

The buoyant force on an object depends only on that portion of the object's volume that is submerged, or "below the waterline" (if the fluid is water). We shall refer to this as the *submerged volume* or equivalently the **displaced volume**, denoted by the symbol $V_{\text{displaced}}$. The "weight of the displaced fluid" refers to the fluid that would occupy this volume if the object were not present. Since the volume is not actually occupied by the fluid, we call it the **displaced fluid**. Hence the upward **buoyant force**, or equivalently, the **weight of the displaced fluid** is:

$$B = \rho_{\text{fluid}} \cdot V_{\text{displaced}} \cdot g \tag{62}$$

The downward force of gravity (i.e. *weight*) on the object is

$$F_{gravity} = \rho_{\text{object}} \cdot V_{\text{object}} \cdot g \tag{63}$$

The net upward force is

$$F_{\text{net up}} = B - F_{gravity} \tag{64}$$

7.4 Continuity Equation for Incompressible Fluids

Consider a pipe with cross-sectional area A that varies along the pipe. If the variation is sufficently smooth, the volume of pipe is $\Delta V = A\Delta x$ along a distance Δx. If this volume passes a point along the pipe in a time interval Δt, then the the speed of the fluid is $v = \Delta x/\Delta t$. The *volume flow rate*, $I_V = \Delta V/\Delta t$ is given by,

$$I_V = Av = \text{constant} \tag{65}$$

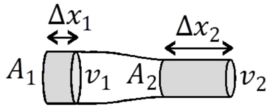

Figure 18: Conservation of volume flow rate for incompressible fluid.

7.5 The Bernoulli Equation

Bernoulli's equation holds for any incompressible fluid when fluid *viscosity* can be neglected. Consider a small cube inside a fluid (i.e., a *fluid element*). Let the length of this cube $\Delta\ell$ be much smaller than the dimensions of any pipe or other structure. Using the concepts developed at (57) and (58), we make the following observations about this fluid element's motion in the $x-$direction:

$$M = \rho \cdot \Delta\ell^3 \tag{66a}$$

$$F_g = \rho g_x \cdot \Delta\ell^3 \tag{66b}$$

$$F_P = [P(x) - P(x+\Delta\ell)] \cdot \Delta\ell^2 \tag{66c}$$

where F_g is the force of gravity, and F_P is the force of pressure (F_P is calculated by subtracting the opposing forces on opposite faces on the cube). Equation (66c) can be mapulated using the definition of "derivative":

$$\lim_{\Delta\ell\to 0} \frac{P(x) - P(x+\Delta\ell)}{\Delta\ell} = -\frac{\partial P}{\partial x} \tag{67}$$

The use of the rounded 'd' (∂) to denote *partial differentiation* because pressure is typically a function of more than one variable. Though a full analysis is beyond the scope of this work, certain concepts can be discerned from casual inspection of (66):

1. The only two forces that act are gravity and pressure
2. Pressure acts as a potential analogous to gravitational potential energy

In contrast to gravity, where a large height implies large potential energy, large pressure corresponds to **low** potential energy. Hence fluids *speed up* at low pressure. The analog to energy conservation is Bernoulli's equation:

$$P + \rho g h + \frac{1}{2}\rho v^2 = \text{constant} \tag{68}$$

8 Statistical Mechanics

8.1 The Ergodoc Hypothesis

Statistical mechanics is based on the premise that the laws of probability can impose predictability on certain systems when the number of particles is extremely large. Central to this approach is to establish how to most conveniently "count" events. For example when flipping a coin twice, it is best to count four events: $\{HH, HT, TH, TT\}$, where $H =$ heads and $T =$ tails. Only by counting TH and HT as separate and **equally probable** events, do we obtain the correct probabilities (25%, 50%, 25%) for obtaining zero, one, or two "heads", respectively. The **ergodic hypothesis** states that once *equally probable* states are identified, a complex system will randomly explore all these states.

For simplicity we develop ideas with unphysical models. Figure 19 shows two circles that contain a two-dimensional ideal gas so rarefied that elastic collisions occur only with smooth walls.

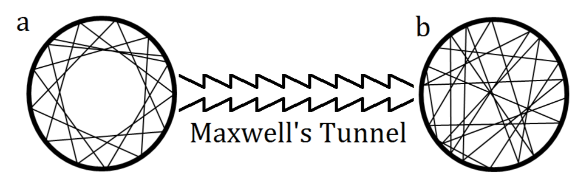

Figure 19: Atoms bouncing off the walls of a container.

If the walls are perfectly smooth, the atoms could theoretically follow a path such as shown in Figure 19 a, leaving a highly improbable vacuum at the center. The ergodic hypothesis implies that atoms move with a random selection of paths, as shown in Figure 19 b. Here, either location, or the angle with respect to any reference line can be viewed as *equally probable*.

To appreciate the significance of the ergodic hypothesis, imagine that it could be violated in the systems of Figure 19 using "Maxwell's tunnel", which is known to induce fruit flies to migrate towards the right. Begin with two containers with ordinary (randomized) air at the same temperature and pressure. Seal both containers and connect with the tunnel. If the balance of random exchange of particles between the two containers could be violated with a cleverly shaped "tunnel", one could construct an engine that extracts the random energy of atoms in a gas and does work (as defined at (30b)). The widely held belief that such a device is impossible is known as the **second law of thermodynamics**.

8.2 Phase Space

Consider a single undriven harmonic oscillator can be modeled by (45) with the driving term F_d set to zero. In phase space, the position and momentum of the particle are plotted as shown in Figure 20. The black quadilateral shape can be viewed from a perspective of probability theory as the range of possible locations in phase space for a single oscillator. Equivalently, one can imagine a large number of noninteracting oscillators and view the shape as a densly packed "cluster" of these oscillators in phase space. The arrows show the motion of a single cluster in phase space as time evolves.

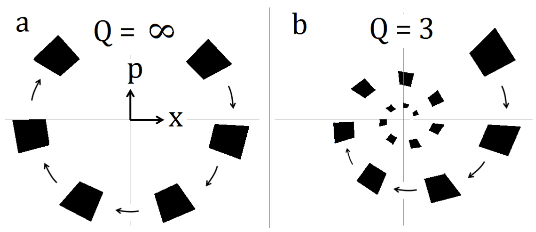

Figure 20: The motion of a "cluster" of orbits in (a) undamped and (b) damped oscillators

If damping is present the "cluster" spirals towards a single point at the origin that represents no motion as shown in Figure 20b. Note that the area (in phase space) also shrinks to zero. If no damping is present (i.e., $Q = \infty$) the **area in phase space is preserved** as time evolves. This is a general result for the motion of objects governed by Hamilton's equations of motion (56a). In general, the shape in phase space is not preserved as time evolves, as illustrated in Figure 21.

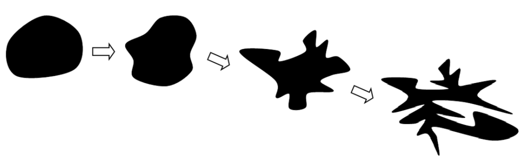

Figure 21: Area, not shape is preserved

Liouville's Theorem states that the area in phase space is invariant whenever motion is governed by Hamilton's equations motion. The proof requires concepts not yet developed, so easily analyzed examples are offered: (1) If no forces are acting on a particle, a rectangle in phase space will distort into a parallegram of the same area. (2) A sharp rise in potential energy between regions of zero force can split a rectangle in phase space between the part of the area with sufficient energy to overcome the potential, and that part of the area that gets reflected. (3) Figure 22 depicts uniform acceleration.

Liouville's thorem also holds for a system of N particles, where a point in phase space is, $(\vec{r}_1, \vec{p}_1, \vec{r}_2, \vec{p}_2 \ldots, \vec{r}_N, \vec{p}_N) = (x_1, y_1, p_{x,1}, p_{y,1}, x_2, y_2, p_{x,2} \ldots)$. Here "area" is actually a volume: $\mathrm{d}x_1 \cdot \mathrm{d}p_{x,1} \cdot \mathrm{d}x_2 \cdot \mathrm{d}p_{x,2} \ldots \mathrm{d}x_N \cdot \mathrm{d}p_{x,N})$.

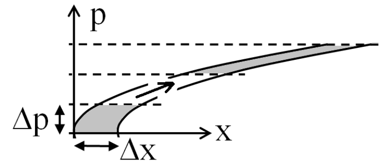

Figure 22: Uniform acceleration from rest

8.3 Probability Distribution Functions

1. If a probability distribution function $f(\xi)$ is evaluated at ξ, then $f\Delta\xi$ is the probability that the argument of the function is between ξ and $\xi+\Delta\xi$. The probability integrates must integrate unity, $\int f(\xi)\mathrm{d}\xi = 1$, where the limits are over the range where $f(\xi) > 0$.

2. Average values involve integrals over $f(\xi)$. Using brackets to denote average: $\langle\xi\rangle = \int \xi f(\xi)\mathrm{d}\xi$, or $\langle\xi^2\rangle = \int \xi^2 f(\xi)\mathrm{d}\xi$, etc.

3. If we desire the distribution function for a different variable, e.g., $\chi = \chi(\xi) = \xi^3$, subscripts can help distinguish the different distribution functions: $f_\xi(\xi)$ and $f_\chi(\chi)$. Use, $f_\chi \mathrm{d}\chi = f_\xi \mathrm{d}\xi$ to derive, $f_\chi \mathrm{d}\chi = f_\xi \cdot \mathrm{d}\xi/\mathrm{d}\chi$.

8.4 The Maxwell-Boltzmann Distribution

Consider a one-dimensional gas of N weakly interacting particles the potential, $U(x)$, shown in Figure 23b. The white curves represent contours of constant energy,$E_n = \frac{1}{2}p^2/m + U(x)$, where the index n denotes the particles $(n = 1, 2, \ldots, N)$. In accordance with the ergodic hypothesis, the **probability distribution function** is uniform along these contours. Darker shades represent more highly populated regions of phase space. The interactions between particles are largely ignored, but are neccessary to randomize the orbits and permit statistical analysis.

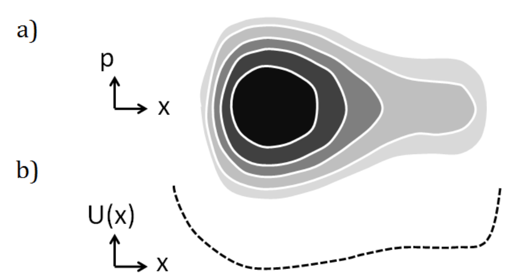

Figure 23: Concentric ergodic regions for a potential well.

Using the weak nature of interactions between particles to postulate **statistical independence**, (13b) implies that the probability distribution can be written as a product:

$$f_{\text{macro}}(x_1, p_1, x_2, p_2, \ldots, x_N, p_N) = f_1(x_1, p_1) \cdot f_2(x_2, p_2) \cdot \cdots \cdot f_N(x_N, p_N) \quad (69)$$

Here f_{macro} is the"macroscopic" probabillity function that describes all N particles in phase space, and is presumed to be a function of the total energy, $E = E_1 + E_2 + \cdots + E_N$, so that $f_{\text{macro}} = f_{\text{macro}}(E_1 + E_2 + \ldots + E_N)$. Phase space for each of the individual particles is described by the "microscopic" probability functions, $f_n = f_n(x_n, p_n) = f_n(E_n)$, as depicted in Figure 23-a. The "microscopic" probability funcitons depend only on the energy, E_n, of a single particle. These dependences on energy for the macro and micro states greatly constrain the two distribution functions. The first identity in (5b) suggests an exponential dependence on energy of the form:

$$f_{\text{grand}} = A^N e^{-\beta(E_1 + E_2 + \ldots + E_N)} \quad (70a)$$

$$f_n(x, p) = Ae^{-\beta E_n} = Ae^{-\beta\left[p^2/(2m) + U(x)\right]} \quad (70b)$$

where A and β are two constants: A can be found by imposing the condition that, $\int f(x, p)\mathrm{d}x\mathrm{d}p = 1$. The other constant can serve as our definition of temperature:

$$\beta = \frac{1}{k_B T}, \quad (71)$$

where $k_B \approx 1.38 \times 10^{-23}$ J/K is Boltzmann's constant. Equation (71) defines temperature on the **Kelvin** scale. If T_C, is measured in the *Celsius*, then $T_C = T - 273.16$.

8.5 3-D Monatomic Ideal Gas

The monatomic ideal gas in three dimensions can be modeled using the methods leading to (70) by setting $U(x) = 0$. It can be shown that the **distribution functions** for velocity, $\vec{v}$, and energy, $E = \frac{1}{2}mv^2$, are,

$$f_{\vec{v}} = \left(\frac{m}{2k_B T}\right)^{3/2} \exp\left(-\frac{mv_x^2 + mv_y^2 + mv_z^2}{2k_B T}\right) \quad (72a)$$

$$f_E = \frac{2}{\sqrt{\pi}}\left(\frac{m}{2\pi k_B T}\right)^{3/2} E^{1/2} \exp\left(\frac{-E}{k_B T}\right). \quad (72b)$$

In the case of (72), we first convert to spherical coordinates (in velocity space) using, $\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z = 4\pi v^2\mathrm{d}v$, where, $v = \sqrt{\vec{v} \cdot \vec{v}} = |\vec{v}|$, is *speed*. Now change variables v to $E = \frac{1}{2}mv^2$, and integrate ($v = 0 \to \infty$) to obtain, $\int f_v v^2 \mathrm{d}v = \langle v^2 \rangle$, which is the average value of v^2.

In a similar fashion we can calculate the total internal energy, E_{int}, as either, $N \cdot \langle E \rangle$, or, $N \cdot \frac{1}{2}mv_{rms}^2$, where N is the number of particles in the ideal gas. For reference we define the **degrees of freedom** for a three-dimensional gas as $\hat{f} = 3$.

$$v_{rms}^2 = \frac{\langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle}{3} \quad (73a)$$

$$E_{\text{int}} = \frac{\hat{f}}{2} N k_B T \quad (73b)$$

The importance of knowing which states are *equally probable* is evident by the extra factor $E^{1/2}$ in (72b). The absence of such a term in (72a) arises from the fact that all values of (v_x, v_y, v_z) are equally probable, as discussed near (69).

8.6 Non-monatomic ideal gasses

Equation (73b) is also valid for one or two dimensional monoatomic ideal gass, if $\hat{f}$ is replaced by the number of dimensions. Except at very low temperatures or high pressures, the noble gasses (H, He, ...) are good approximations to an ideal monoatomic gas. For reasons not adequately modeled by classical physics, air and diatomic gases can be modeled by assuming five degrees of freedom, so that $\hat{f} = 5$ is a good approximation (at moderate temperatures).

8.7 Ideal Gas Law

The pressure, P on a surface is defined as the ratio of the force, F, to area, A:

$$P = F/A, \quad (74)$$

Pressure is caused by the flux of particles striking the surface, as shown in Figure 24. We begin with a simplistic model that assumes all particles travel at the same speed. The **particle flux** is defined as the rate at which particles strike the area, A. The number of particles, ΔN, striking the wall in a time, Δt, equals the number of particles the cylinder shown. This volume equals $A\Delta x$, where $\Delta x = v\Delta t$. The number of particles in this volume is $\Delta N = nA\Delta x$. Therefore,

Figure 24 establishes the concept of **flux**. Particles in a simple gas all have the same speed, directed at an area A. If $n*$ is density of particles moving to the left, then:

$$\text{Particle Flux} = \frac{\Delta N}{\Delta t} = \frac{1}{2}n^* vA. \quad (75)$$

During time, Δt, all the particles in the area shown in Figure 24 bounce off the wall. By (38) the average force

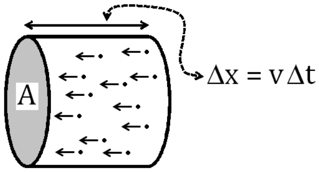

Figure 24: Flux of particles striking a surface

during this time interval is for each atom, and $\frac{1}{2}n^* vA\Delta t$ strike the wall. In our simple model, the density of atoms travelling to the left equals that to the right, so that $n^* = \frac{1}{2}n$.

$$\langle F \rangle = \left\{ \frac{2mv}{\Delta t} \right\} \left\{ \frac{n}{2} vA\Delta t \right\} \tag{76a}$$
$$P = nmv^2 \rightarrow nm\langle v^2 \rangle = nk_B T \tag{76b}$$

where we used (73a) to deduce that $mv_x^2 = k_B T$, a result that applies for any ideal gas, regardless of the value of $\hat{f}$. Since $n = N/V$ is density, $PV = Nk_B T$.

9 Thermodynamics

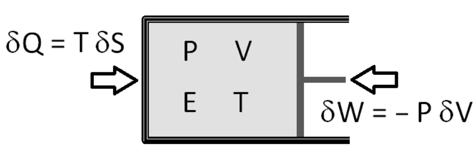

Figure 25: An isolated system with four state variables

Consider a sealed cylinder with a movable piston, as shown in Figure 25. If the system is at **equilibrium** there are four **state variables**, (P, V, E, T). Changes in these variables are assumed to made slowly enough that the approximation of **quasi-static** equilibrium is valid.

The flow of energy due to a difference in temperature is called **heat**. We shall denote a quantity of heat that flows *into* the system as, $\mathrm{d}Q$. Energy can also be added by doing work *on* the system, $dW = -F\mathrm{d}x = -P\mathrm{d}V$, where F is the force on piston of area A as it moves a distance $\mathrm{d}x$. The following statement of energy conservation is known as the **first law of thermodynamics**:

$$\mathrm{d}E = \mathrm{d}Q - P\mathrm{d}V = \mathrm{d}Q + \mathrm{d}W \tag{77}$$

Heat is the flow of energy between substances of different temperature (always from *hot* to *cold*). The **heat capacity**, C_*, can be defined as the amount of heat required to raise the temperature of an object by one degree. In complex substances, heat can flow in or out of a system

$P(V,T) = Nk_BT/V$	$E(V,T) = \hat{f}Nk_BT/2$	(a)
$P(V,E) = 2E/(\hat{f}V)$	$T(V,E) = 2E/(\hat{f}Nk_B)$	(b)
$V(P,T) = Nk_BT/P$	$E(P,T) = \hat{f}Nk_BT/2$	(c)
$V(P,E) = 2E/(\hat{f}P)$	$T(P,E) = 2E/(\hat{f}Nk_B)$	(d)
$E(P,V) = \hat{f}PV/2$	$T(P,V) = PV/(Nk_B)$	(e)

Table 2: State functions and variables for the ideal gas. The number of degrees of freedom is $\hat{f} = 2/(\gamma - 1)$

without changing temperature. What changes instead is the chemical **phase** of the substance (e.g. ice to water, or water to steam). The coefficient of **latent heat**, L, is defined as the amount heat required to change the phase of a one kilogram of a substance. It is an **intrinsic** property of matter, meaning that L does not depend on the amount present. For our purposes the **extrinsic** form of heat capacity, C_*, is more convenient:

$$L = \frac{1}{\text{mass}} \frac{\delta Q}{\delta T} \tag{78a}$$
$$C_* = \frac{\delta Q}{\delta T} = \frac{\delta E}{\delta T} + P\frac{\delta V}{\delta T} \tag{78b}$$

Here, (77) has been used to rewrite the expression for C_*. The asterisk on C_* reminds us that heat capacity is not yet fully defined because we must also stipulate whether volume or pressure is held constant during the process. If $\delta V = 0$, (78b) defines the heat capacity at constant volume, C_V. If $\delta P = 0$, (78b) defines the heat capacity at constant pressure, C_P.

9.1 Ideal Gas Revisited

Our four state variables (P, V, E, T) are interrelated by the two equations (73b) and (76b). Since energy depends only on temperature, there are five ways to express two of the four variables as functions of the other two. These are displayed in Table 2, which can be used to calculate partial derivatives. For example, from (c) we have: $\partial V(P,T)/\partial T = Nk_B/P$. It is convenient notation is to use a vertical bar to denote what variables are held constant, so that, $\partial V/\partial T\big|_P$ denotes $\partial V/\partial T$ when $V = V(P,T)$.

The relation between E and T for an ideal gas, that the first term on the RHS of (78b) is $\partial E/\partial T = \hat{f}Nk_B$, regardless of whether P or V is held constant. If pressure is held constant, the second term on the RHS can be found from $V(P,T)$ in Table 2c where it is shown that $\delta V = \partial V/\partial T\big|_P \delta T = (Nk_B/P)\delta T$. Hence,

$$C_V = \frac{1}{2}\hat{f}Nk_B = \frac{1}{\gamma - 1}Nk_B, \tag{79a}$$

$$C_P = C_V + Nk_B = \gamma C_V, \tag{79b}$$

$$\text{where } \gamma = \frac{\hat{f}+2}{\hat{f}}, \tag{79c}$$

is called the **adiabatic index**.

9.2 Entropy

We define **entropy**, $S = S(E,V)$, through its differential.

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T} = \frac{\mathrm{d}E}{T} + \frac{P\mathrm{d}V}{T}, \tag{80a}$$

$$= C_V\frac{\mathrm{d}T}{T} + Nk_B\frac{\mathrm{d}V}{V} \tag{80b}$$

$$\rightarrow S = C_V \ln\frac{T}{T_0} + Nk_B \ln\frac{V}{V_0}. \tag{80c}$$

Equation (80a) is the general form for entropy. Equations (80b)-(80c) are restricted to an ideal gas, using Table (2-c) to express E and P in terms of V. T_0 and V_0 are reference parameters associated with an arbitrary constant of integration.

9.3 Isothermal and Adiabatic Processes

An isothermal process occurs at constant temperature, $T_\mathrm{i} = T_\mathrm{f}$, which by the ideal gas law, implies, $P_\mathrm{i}V_\mathrm{i} = P_\mathrm{f}V_\mathrm{f}$. Simple substitution, $\int P\mathrm{d}V = P_\mathrm{i}V_\mathrm{i}\int \mathrm{d}V/V$, leads to a logarithmic expression involving the work associated with an isothermal process in an ideal gas, $-\mathrm{d}W = \int P\mathrm{d}V$:

$$\int_{V_\mathrm{i}}^{V_\mathrm{f}} PdV = nRT\ln\frac{V_\mathrm{f}}{V_\mathrm{i}}. \tag{81}$$

If no heat flows in or out of an ideal gas, $\mathrm{d}Q = 0$, and the process is **adiabatic**. Substituting $\mathrm{d}E = C_V dT$ and $P = Nk_BT/V$ into (77) yields for adiabatic processes in an ideal gas,

$$C_V\mathrm{d}T + \frac{Nk_BT}{V}\mathrm{d}V = 0, \tag{82a}$$

$$\frac{\mathrm{d}T}{T} + (\gamma - 1)\frac{\mathrm{d}V}{V} = 0, \tag{82b}$$

where we have used (79) to simplify (82a). Integrate (82b) and use (5) to simplify the logarithms. Expressed in terms of P and V we have for adiabatic processes in an ideal gas:

$$P_\mathrm{f}V_\mathrm{f}^\gamma = P_\mathrm{i}V_\mathrm{i}^\gamma \tag{83a}$$

$$\int_{V_\mathrm{i}}^{V_\mathrm{f}} PdV = C_V\left(T_\mathrm{i} - T_\mathrm{f}\right) \tag{83b}$$

Equation (83b) could be calculated by integration using (83a), but is more easily found by using the change in energy since $\int PdV = -\Delta E$ if $\mathrm{d}Q = 0$.

Equation (80a) is valid for any (classical) thermodynamic system, while (80c-d) hold only for the ideal gas.

9.4 Carnot Cycle with an Ideal Gas

A **heat engine** is a device that moves heat from a *heat bath*, or **reservoir**, at high temperature to a reservoir at low temperature, doing work in the process. A **heat pump** operates in reverse by using work to transfer heat from a low temperature reservoir to one at high temperature. We shall consider only a simple valveless heat engine that take the piston of Figure 25 through a cycle of heating and cooling. The most important heat engine follows a path discovered by Carnot in 1824, which sketched in the pressure versus volume (P-V) curve in Figure 26. (This graph is not to scale.) The path 1-2-3-4-1 is a heat engine, while the reverse path serves as a cooling unit (i.e. air conditioner). The net work done during one cycle W_cycle can be expressed using the closed line integral introduced after (31c):

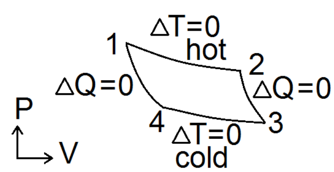

Figure 26: Carnot cycle (schematically drawn)

$$W_\mathrm{cycle} = \int_{V_1}^{V_3} \overbrace{\mathrm{P}dV}^{123} + \int_{V_3}^{V_1} \overbrace{\mathrm{P}dV}^{341} \tag{84a}$$

$$\equiv \oint \mathrm{P}dV \tag{84b}$$

the *overbrackets* indicate which path is taken. No minus sign is required in (84a) because the limits guarantee that the second term is negative.

Since W_cycle is defined as work done *by* the gas and not *on* the gas, and since the net change in energy (over one cycle) is zero, we can use (77) to write:

$$W_\mathrm{cycle} = \oint \mathrm{d}Q = Q_\mathrm{hot} - Q_\mathrm{cold}, \tag{85}$$

where Q_{hot} is the heat that flows from the high temperature reservoir into the gas, and Q_{cold} is the **waste heat** that flows into the low temperature reservoir. A useful measure of a heat engine's efficiency, η, is:

$$\eta = \frac{W_{\text{cycle}}}{Q_{\text{hot}}} = \frac{Q_{\text{hot}} - Q_{\text{cold}}}{Q_{\text{hot}}} = 1 - \frac{Q_{\text{cold}}}{Q_{\text{hot}}}. \tag{86}$$

The efficiency of a Carnot cycle can be calculated using entropy. The two paths 12 and 34 in Figure 26 are **isotherms**, while the paths 23 and 41 are **adiabats**. By (80a) entropy does not change along an adiabat (since $\delta Q = 0$). Define S_{14} as the entropy along path 14 and S_{23} as the entropy along path 23. Define the difference in entropy between the adiabats as, $\Delta S = S_{23} - S_{14}$. From (80c), we have, $Q_{\text{hot}} = T_{\text{hot}}\Delta S$, and, $Q_{\text{cold}} = T_{\text{cold}}\Delta S$. Hence,

$$\frac{T_{\text{cold}}}{T_{\text{hot}}} = \frac{Q_{\text{cold}}}{Q_{\text{hot}}}, \tag{87a}$$

$$\eta = 1 - \frac{T_{\text{cold}}}{T_{\text{hot}}}. \tag{87b}$$

9.5 Carnot Cycle with Other Working Fluids

The student may have wondered how the Carnot engine could exchange heat with either reservoir (T_{hot} or T_{cold}) if the working fluid and reservoir are at the same temperature. This is accomplished only in approximation; in reality there must be a small temperature change. This peciluar property of the Carnot cycle also guarantees **reversibility**: Any flow of heat that occurs in the Carnot cycle can be "reversed" (in approximation) by interchanging temperature of the working fluid between *slightly below* and *slightly above* that of the reservoir. Our analysis of heat engines ignores all **irreversible** proccesses. Irreversible processes include friction in the piston mechanism, and heat flow between objects at different temperatures. A sudden expansion or compression of the working fluid is also irreversible.

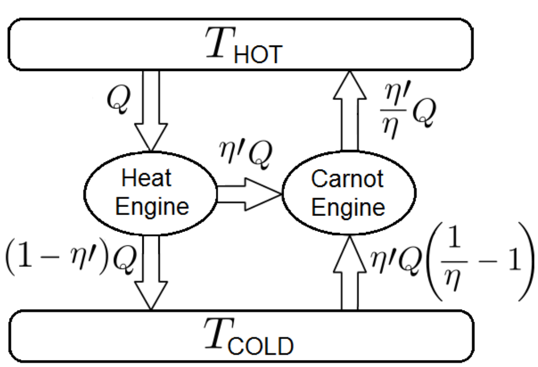

Figure 27: Proof of Carnot's Theorem

Figure 27 shows a heat engine of efficiency η' that is driving a Carnot engine with an ideal gas serving as the working fluid. In this figure, the carnot engine is operating as a heat pump, but since this cycle is reversible, all ratios invoving work per cycle (W_{cycle}) and heat per cycle (Q_{hot}) and (Q_{cold}) are the same. Hence the efficiency of the ideal-gas Carnot engine serving as a heat pump in the figure is $\eta = 1 - T_{\text{cold}}/T_{\text{cold}}$, as we have already established.

We now show that, $\eta' = \eta$, where is the efficiency of the second heat engine shown in the figure. Both engines are assumed to be irreversible (i.e. "perfect") heat engines. We argue first that $\eta' \leq \eta$. Otherwise the combined engines of Figure 27 would pump heat from the cold reserviour to the hot one without requiring work input, in violation of the **second law of thermodynamic**. But if $\eta' < \eta$, one could reverse the working fluids and also create a device that pumps heat from the hot into the cold reservoirs without doing work. Hence we conclude that for any working fluid, the efficiency of any reversible engine is given by (87b).

It is interesting to note that (80a) was integrated to form the state function entropy, $S = S(T, V)$, only for the ideal gas. Not all differentials involving two or more variables can be integrated, with heat, dQ, being the most notable counter-example. But the argument associated with Figure 27 suggests that entropy exists for any substance capable of reversible change. As was the case with our "proof" of **Newton's second law**, our "proof" that classical entropy can be integrated is logically flawed because the **second law of thermodynamics** was asserted without proof to support a claim that "Maxwell's tunnel" of Figure 19 would not move particles from a region of low density to one of high density. Physics is not held together by tight logic.