Appendix: Proofs to Mathematical Propositions in "Job Release Policy and Printed Circuit Board Assembly" by M.J. Magazine and G.G. Polak, *IIE Transactions*, 2003

Proof of Theorem 1.

Separating the minimizations in (4) yields

$$z^* = \min_{\boldsymbol{s} \in \mathbf{F}^K, w \in Z^+, \boldsymbol{m} \in W(K,w)} \left\{ \sum_{q=1}^{w} \left(\sigma + \sum_{\{k: m_q \le \rho(k) < m_{q+1}\}} \sum_{i \in \mathbf{N}} b^k r^k(i) d[s^q(i)] \right) \right\}$$

$$= \min_{w \in Z^+, \boldsymbol{m} \in W(K,w)} \left\{ \sum_{q=1}^{w} \left[\sigma + \min_{\boldsymbol{s} \in \mathbf{F}^K} \sum_{\{k: m_q \le \rho(k) < m_{q+1}\}} \sum_{i \in \mathbf{N}} b^k r^k(i) d[s^q(i)] \right] \right\}$$

$$= \min_{w \in Z^+, \boldsymbol{m} \in W(K,w)} \left\{ w\sigma + \wp\left(\left\{k : m_q \le k < m_{q+1}\right\}\right) \right\}$$

where $\wp\left(\left\{k : m_q \le k < m_{q+1}\right\}\right) = \min_{\boldsymbol{s} \in \mathbf{F}^K} \left\{ \sum_{\{k: m_q \le k < m_{q+1}\}} \sum_{i \in \mathbf{N}} b^k r^k(i) d[s^q(i)] \right\}$ as defined as in (2) to be the minimal processing time for the q^{th} cluster of jobs, and is independent of the partition of **K** into clusters. Thus the cluster setup (2) emerges as an inner minimization performed over assignments within (4).

Now consider the auxiliary network described in Section 4.2: there is a node for each job *k*, as well as a dummy terminal node (*K*+1) that follows all others. There is an arc for every pair of nodes (*j*,*k*) such that job *k* follows job *j* in the given processing sequence. Each arc (*j*,*k*) has length $\sigma + \wp\left(\{ j, \ldots, k-1\}\right)$ that is, the sum of the setup and the optimal organ pipe objective (2) for the cluster consisting of jobs *j* through (*k*-1). Given a partition of the set of jobs **K** into clusters each of which is a subset of consecutive jobs in the release sequence, it follows that total manufacturing time is the length of a path from node 1 to node (K+1) along the arcs corresponding to these clusters.

Moreover, because this network is acyclic and has $(K+1)\cdot(K)/2$ arcs, a shortest path can be found in $O(K^2)$ time by the reaching algorithm given by Ahuja, Magnanti and Orlin (1993). ♦

Proof of the Lemma (by contradiction). First observe that by searching over both the subsequence breakpoints in ***m*** and the permutation sequence h for each term in (14) for a fixed number of clusters, we are minimizing processing time over all partitions of **K** into so many clusters. Now consider any integer ν, 1≤ν≤K-1, and suppose contrary to (14) that

$$\min_{\boldsymbol{m}\in\mathbf{W}(K,\nu),h\in\mathbf{H}} \sum_{q=1}^{\nu} \wp(C_{\rho_h(q)}) < \min_{\boldsymbol{m}\in\mathbf{W}(K,\nu+1),h\in\mathbf{H}} \sum_{q=1}^{\nu+1} \wp(C_{\rho_h(q)}), \quad (18)$$

that is, the minimal processing time over all partitions of **K** into ν clusters, occurring, say, for a partition $\Re$ is strictly less than the minimal processing time over all partitions of **K** into (ν*+1)* clusters. There must exist a cluster in $\Re$ that can be split into two subclusters, yielding a partition $\Re'$ of **K** into (ν*+1)* clusters. Moreover, the total processing time for $\Re'$ can be no more than that of $\Re$ by superadditivity (3). But the minimum on the right hand side of (18) is taken over all clusters of size (ν*+1)*. Therefore, the fact that processing time for $\Re'$, with (ν+1) clusters, is no less than the left hand side of (18) leads to a contradiction.

Now we must prove that the first and last inequalities hold in the strict sense. Suppose to the contrary that

$$\min_{\boldsymbol{m}\in\mathbf{W}(K,K),h\in\mathbf{H}} \sum_{q=1}^{K} \wp(C_{\rho_h(q)}) = \min_{\boldsymbol{m}\in\mathbf{W}(K,K-1),h\in\mathbf{H}} \sum_{q=1}^{K-1} \wp(C_{\rho_h(q)}) \qquad (19)$$

Observe that forming K-1 clusters from K singleton clusters can only be accomplished by combining two of the singletons. But (19) implies that no such pairing exists to decrease the processing time of K singletons; thus all jobs must have identical organ pipe setups, contrary to the stated assumptions. Finally consider the other end of the series; if

$$\min_{\boldsymbol{m}\in\mathbf{W}(K,2),h\in\mathbf{H}} \sum_{q=1}^{2} \wp(C_{\rho_h(q)}) = \wp(\{1,\ldots,K\})$$

then no splitting of the grand inclusive cluster exists to decrease total processing time. However, this would also imply that all jobs have identical organ pipe setups, contrary to the stated assumptions. ♦

Proof of Theorem 2.

First, we establish existence for σ^{lower} and σ^{upper}. For σ=0 total manufacturing time consists solely of processing time. By superadditivity (3), minimal processing time occurs for w^* =K clusters. Thus, we have demonstrated the existence of $\sigma^{\text{lower}} \geq 0$.

Next consider a setup time equal to the time needed to process all jobs grouped in a common cluster, $\tilde{\sigma} = \sum_{k=1}^{K} \left[\sum_{i\in\mathbf{N}} b^k r^k(i) d[s^1(i)] \right]$, where $\mathbf{s}^1$ denotes some feasible assignment of components to sleeves for this common cluster. Because the additional time required to setup for more than one cluster would exceed any possible reduction in processing time due to creating more than one cluster, we have demonstrated the existence of $\sigma^{upper} \le \tilde{\sigma}$.

Moreover, the strict inequality $\sigma^{lower} < \sigma^{upper}$ must hold for K>2. Otherwise, $\sigma^{lower} = \sigma^{upper}$ would imply that the total manufacturing time would be the same for any number of clusters from 1 through *K*, inclusive, i.e.,

$$
\begin{aligned}
& K \cdot \sigma + \min_{\boldsymbol{m} \in \mathbf{W}(K,K), h \in \mathbf{H}} \sum_{q=1}^{K} \wp(C_{\rho_h(q)}) \\
& = (K-1) \cdot \sigma + \min_{\boldsymbol{m} \in \mathbf{W}(K,K-1), h \in \mathbf{H}} \sum_{q=1}^{K-1} \wp(C_{\rho_h(q)}) \\
& = \cdots = 2 \cdot \sigma + \min_{\boldsymbol{m} \in \mathbf{W}(K,2), h \in \mathbf{H}} \sum_{q=1}^{2} \wp(C_{\rho_h(q)}) \\
& = \sigma + \wp(\{1, \ldots, K\})
\end{aligned}
$$

However, such an increase in processing time, i.e. linear in σ to exactly offset the decrease in setup time, would violate superadditivity (3).

Now we can determine σ^{lower} and σ^{upper} by doing parametric calculations based on the minimal processing times employed in Lemma 2. Separating the searches in (5) for any value of σ≥0 yields

$$
\begin{aligned}
z^* &= \min_{\boldsymbol{s} \in \mathbf{F}^K, w \in Z^+, \boldsymbol{m} \in W(K,w), h \in \mathbf{H}} \left\{ \sum_{q=1}^{w} \sum_{\{k: m_q \le \rho_h(k) < m_{q+1}\}} \left[\sigma + \sum_{i \in \mathbf{N}} b^k r^k(i) d[s^q(i)] \right] \right\} \\
&= \min_{w \in Z^+} \left\{ w \cdot \sigma + \sum_{q=1}^{w} \min_{\boldsymbol{m} \in W(K,w), h \in \mathbf{H}} \left[\sum_{\{k: m_q \le \rho_h(k) < m_{q+1}\}} \min_{s_q \in \mathbf{F}^K} \left(\sum_{i \in \mathbf{N}} b^k r^k(i) d[s^q(i)] \right) \right] \right\} \\
&= \min_{w \in Z^+} \left\{ w \cdot \sigma + \sum_{q=1}^{w} \min_{\boldsymbol{m} \in W(K,w), h \in \mathbf{H}} \wp(C_{\rho_h(q)}) \right\}
\end{aligned}
$$

For σ=0, the optimal number of clusters is $w^* = K$, but we know from the Lemma that there exists some setup time σ>0 for which $w^* = K-1$. Thus, finding the maximal value of σ for which

$$K \cdot \sigma + \min_{m \in W(K,K), h \in \mathbf{H}} \sum_{q=1}^{K} \wp(C_{\rho_h(q)})$$

$$\leq (K-1) \cdot \sigma + \min_{m \in W(K,K-1), h \in \mathbf{H}} \sum_{q=1}^{K-1} \wp(C_{\rho_h(q)})$$

is true yields (12). Similarly, for $\sigma = \tilde{\sigma}$, the optimal number of clusters is $w^* = 1$, but that for some setup time $\sigma < \tilde{\sigma}$, $w^* = 2$. Thus, finding the minimal value of σ for which

$$\sigma + \wp(\{1, \ldots K\}) \leq 2 \cdot \sigma + \min_{\boldsymbol{m} \in W(K,2), h \in \mathbf{H}} \sum_{q=1}^{2} \wp(C_{\rho_h(q)})$$

holds in turn yields (13).

Finally, the posited relationship between optimal objective values $z^{\text{CP1}}(\sigma) \leq z^{\text{FSO}}(\sigma)$ for all $\sigma>0$ must hold simply because CP1 minimizes an objective function identical to that of FSO, but over a larger feasible set