

Ch 19 – 2-Factor Studies—Equal Sample Sizes

Factorial Experiments:

What are they? Why are they useful?

Example: Castle Bakery Company (p833)

A 2-replicate, 3×2 factorial experiment in a completely randomized design (CRD)

Factors and notation: factors “A” and “B” (or equivalently, “height” and “width”), with a and b levels, respectively.

Advantages of factorial experiments

Simple/naive approach to data analysis: analyze like a single factor design

Cell means or treatment effects model, $Y_{ijk} = \mu + \tau_{ij} + \epsilon_{ijk}$, with LSEs $\hat{\mu} + \hat{\tau}_{ij} = \bar{Y}_{ij.}$.

ANOVA, comparison of effects, and diagnostics and remedial measures the same as if one factor with $r = ab$ levels.

Example: ch19eg1.txt: checking model assumptions, ANOVA, Tukey’s method

Preferred/standard analysis: take into account the factorial treatment structure by modeling the *factorial effects*, including *main effects* and *interactions*, and analyzing these.

Model (19.23), p832, but without constraints:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}, \quad \epsilon_{ijk} \sim N(0, \sigma^2) \text{ and independent}$$

Discuss plot of $Y * A = B$, (SAS p6). What are main effects? What are interactions, or interaction effects?

ANOVA: build up the model sequentially using general linear tests.

Example: include A, then B, then A*B

Model	SSE(R) – SSE(F)	Δ d.f.
$Y_{ijk} = \mu + \epsilon_{ijk}$	$\sum_{ijk} (Y_{ijk} - \bar{Y}_{...})^2 - \sum_{ijk} (Y_{ijk} - \bar{Y}_{i..})^2$ $= \dots = bn \sum_i (\bar{Y}_{i..} - \bar{Y}_{...})^2 = \text{SSA, say}$	
$Y_{ijk} = \mu + \alpha_i + \epsilon_{ijk}$	$\sum_{ijk} (Y_{ijk} - \bar{Y}_{i..})^2 - \sum_{ijk} [Y_{ijk} - (\bar{Y}_{i..} + \bar{Y}_{.j.} - \bar{Y}_{...})]^2$ $= \dots = an \sum_j (\bar{Y}_{.j.} - \bar{Y}_{...})^2 = \text{SSB, say}$	
$Y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk}$	$\sum_{ijk} [Y_{ijk} - (\bar{Y}_{i..} + \bar{Y}_{.j.} - \bar{Y}_{...})]^2 - \sum_{ijk} (Y_{ijk} - \bar{Y}_{ij.})^2$ $= \dots = n \sum_{ij} (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2 = \text{SSAB, say}$	
$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$		

Comments:

- See the ANOVA Table 19.8, p841
- Compute change in d.f. between models
- The third model, $Y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk}$, is called the *main effects* model, and the fitted values $\hat{Y}_{ijk} = \bar{Y}_{...} + (\bar{Y}_{i..} - \bar{Y}_{...}) + (\bar{Y}_{.j.} - \bar{Y}_{...}) = \bar{Y}_{i..} + \bar{Y}_{.j.} - \bar{Y}_{...}$ are for equal sample sizes only
- SSB (=SSB|A) and SSAB (=SSAB|A,B) have these simple forms only for equal sample sizes
- 3 tests (for equal sample sizes). Some prefer to test interactions first.
- *Interaction effects*: Discuss data plots and means plots, (sometimes called ‘interaction plots’)
- (See chapter 23 for unequal sample sizes)

For the CBC data (p833):

- Do a formal data analysis (FDA), i.e. determine questions/hypotheses and analysis up front
- Do an exploratory data analysis (EDA)
- Discuss “Strategy for Analysis” flowchart, p848
 - Main effects may be important/or interest whether or not interactions are present/significant
 - for FDA, ask up front what questions are relevant!!!

Pooling of Sums of Squares. What is it? *Don't do it!*

Kimball Inequality

The ANOVA involves doing 3 tests, and one could control the simultaneous error rate using the Bonferroni inequality, giving $\alpha \leq \alpha_A + \alpha_B + \alpha_{AB}$.

One can do better using *Kimball's inequality*, since the 3 F -statistics have independent numerators and the same denominator, one can show that one gets as a bound the simultaneous error rate one would get if the three tests were independent. It follows that $\alpha \leq 1 - (1 - \alpha_A)(1 - \alpha_B)(1 - \alpha_{AB})$.

Example: Compare these two inequalities if each test uses a 5% level of significance.

Comparing Treatment Effects. Individual CIs and tests utilize the t -distribution.

Notation: let $\mu_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}$. Then $\hat{\mu}_{ij} = \bar{Y}_{ij.}$, whether or not sample sizes are equal.

For any treatment contrast $L = \sum_{ij} c_{ij} \mu_{ij}$ ($\sum_{ij} c_{ij} = 0$), the LSE is $\hat{L} = \sum_{ij} c_{ij} \bar{Y}_{ij.}$. Also, $S^2(\hat{L}) = \hat{\sigma}^2 \sum_i c_{ij}^2 / n_{ij}$, which reduces to $S^2(\hat{L}) = (\hat{\sigma}^2 / n) (\sum_i c_{ij}^2)$ if samples sizes are equal with common sample size n .

Any treatment contrast for which the coefficients c_{ij} only depends on i is called a *main-effect-of-A contrast*. WLOG, these can be written in the form $L_A = \sum_i c_i \bar{\mu}_{i.}$. If sample sizes are equal, then the least squares estimate is $\hat{L}_A = \sum_i c_i \bar{Y}_{i..}$, with $S^2(\hat{L}_A) = (\hat{\sigma}^2 / bn) (\sum_i c_i^2)$. The main effect of A is composed of such contrasts.

Similarly, any treatment contrast for which the coefficients c_{ij} only depends on j is called a *main-effect-of-B contrast*. WLOG, these can be written in the form $L_B = \sum_j c_j \bar{\mu}_{.j.}$. If sample sizes are equal, then the least squares estimate is $\hat{L}_B = \sum_j c_j \bar{Y}_{.j.}$, with $S^2(\hat{L}_B) = (\hat{\sigma}^2 / an) (\sum_j c_j^2)$. The main effect of B is composed of such contrasts.

Any treatment contrast $L = \sum_{ij} c_{ij} \mu_{ij} = \mathbf{c}'\boldsymbol{\mu}$ for which the coefficient vector $\mathbf{c}_{1 \times ab}$ is orthogonal to all main effect contrast vectors (and the vector of all ones, by definition) is called an *AB-interaction contrast*, or simply, an *interaction contrast*. The AB interaction is composed of such contrasts.

Multiple Comparisons. Whether or not sample sizes are equal:

Tukey's method can be used to obtain simultaneous confidence intervals for: (i) all pairwise comparisons among the $r = ab$ treatment combinations, using $T = q(1 - \alpha; ab, \nu)/\sqrt{2}$; (ii) all pairwise comparisons of levels of factor A, using $T = q(1 - \alpha; a, \nu)/\sqrt{2}$; or (iii) all pairwise comparisons of levels of factor B, using $T = q(1 - \alpha; b, \nu)/\sqrt{2}$. The main effect comparisons may or may not be of interest when interactions are included in the model. If one applies Tukey's method to compare the levels of each factor pairwise, the Bonferroni method can be used to obtain simultaneous confidence levels.

Scheffé's method can be used for example to obtain simultaneous confidence intervals for: (i) all treatment contrasts, using $S^2 = (ab - 1)F(1 - \alpha; ab - 1, \nu)$; (ii) all main-effect-of-A contrasts, using $S^2 = (a - 1)F(1 - \alpha; a - 1, \nu)$; (iii) all main-effect-of-B contrasts, using $S^2 = (b - 1)F(1 - \alpha; b - 1, \nu)$; or (iv) all main-effect contrasts for either factor, using $S^2 = (a + b - 2)F(1 - \alpha; a + b - 2, \nu)$.

The Bonferroni method can be applied to obtain simultaneous confidence intervals for any set of g pre-planned comparisons.

Main-Effects Model

If one is willing to (has good reason to) assume that factors A and B do not interact, then one can model the data using the *main-effects model*, for which

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk}.$$

ANOVA for the main effects model will not include interaction effects, so the number of error degrees of freedom will be $\nu = ab(n - 1) + (a - 1)(b - 1)$ for equal sample sizes, or more generally $\nu = (n_T - 1) - (a - 1) - (b - 1)$.

Multiple comparisons will generally involve using Tukey's method to compare the levels of one or both factors pairwise, combining the two procedures by the Bonferroni method in the latter case.

Model assumptions are evaluated in the usual way using residual analysis.

19.11 Planning of Sample Sizes for Two-Factor Studies

For testing equality of all ab treatment effects, the methods considered for one-way-ANOVA are still applicable. Modifications are needed for the 2-way ANOVA for testing main effects or interactions.

Power approach For testing main effects or interactions, Table B.11 can be used iteratively to determine the sample size needed to achieve specified power for specified values of the parameters under the alternative hypothesis. See p862 for the necessary non-centrality parameter formulae. Table B.12 can be used to determine the sample size for testing main effects if sample sizes are large enough that error degrees of freedom are large. For example, to compute power for testing main effects of A, use Table B.12 with $r = a$, then the resulting sample size is bn . Similar modifications are needed for main effects of B.

Estimation Approach: This works the same as before, using the confidence interval formula for the method of multiple comparisons being applied, then iteratively determining the required sample size to satisfy the requirement, assuming a value for or bound on MSE.

Homework Project. For your data set: (i) plan a FDA; (ii) do your planned FDA; and (iii) do an EDA. Write up the results in a short report. Be terse about model checking, focusing primarily on your FDA plan and the two data analyses. Use the data from the following exercise: Arjun and Liping, 19.10; Jeremy and Allison, 19.12; Bryan, El and Qian, 19.14; Mohamed and Ryan, 19.16; Dacia and John, 19.18. I may ask you to briefly discuss your analysis the day these are due.