Ch 16 – Single Factor ANOVA for Fixed Effects Model, (aka Model I) (revised 1/13/09)

Consider an experiment to compare the effects of a single factor A at r levels, say using a completely randomized design.

Factor Effects Model:

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij}, \quad i = 1, \ldots, r, \quad j = 1, \ldots, n_i,$$

where μ and τ_i are fixed unknown parameters, and the ϵ_{ij} are i.i.d. $N(0, \sigma^2)$.

(*Cell Means Model (16.2):* let $\mu + \tau_i = \mu_i$)

It follows that $Y_{ij} \sim N(\mu + \tau_i, \sigma^2)$ and independent.

Note: This *factor effects model* is not the same as (16.62), as no constraints on the parameters are imposed here.

The Matrix Model:

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \sim N(\boldsymbol{X}\boldsymbol{\beta}, \boldsymbol{I}\sigma^2),$$

where $\boldsymbol{Y}$, $\boldsymbol{X}$, $\boldsymbol{\beta}$, $\boldsymbol{\epsilon}$ and $\boldsymbol{I}$ are. . .

Notation: $n_T = \sum_i n_i$, $\bar{Y}_{i.} = \sum_j Y_{ij}/n_i$, $\bar{Y}_{..} = \sum_i \sum_j Y_{ij}/n_T$

16.5–6 — ANOVA and F-test for equality of factor level effects (or means)

Want to test $H_o: \ \tau_1 = \cdots = \tau_r$ (or equivalently, $\mu_1 = \cdots = \mu_r$)

Develop the test of hypothesis using the General Linear Test Approach, (Sec. 2.8)

Full model: $Y_{ij} = \mu + \tau_i + \epsilon_{ij}$

$$\text{SSE(F)} = \sum_i \sum_j (Y_{ij} - \hat{\mu} - \hat{\tau}_i)^2$$

$$\begin{aligned} \partial\text{SSE(F)}/\partial\hat{\mu} = 0 \quad &\Rightarrow \quad \textstyle\sum_i \sum_j (Y_{ij} - \hat{\mu} - \hat{\tau}_i) = 0 \\ &\Rightarrow \quad \textstyle\sum_i \sum_j e_{ij} = 0 \end{aligned}$$

$$\begin{aligned} \text{For } i = i', \quad \partial\text{SSE(F)}/\partial\hat{\tau}_{i'} = 0 \quad &\Rightarrow \quad \textstyle\sum_j (Y_{i'j} - \hat{\mu} - \hat{\tau}_{i'}) = 0 \\ &\Rightarrow \quad \hat{\mu} + \hat{\tau}_{i'} = \bar{Y}_{i'.} \end{aligned}$$

So, under the full model, $\text{SSE(F)} = \sum_i \sum_j (Y_{ij} - \bar{Y}_{i.})^2 = \text{SSE}$, say

Reduced model: $Y_{ij} = \mu + \tau + \epsilon_{ij}$

$$\text{SSE(R)} = \sum_i \sum_j (Y_{ij} - \hat{\mu} - \hat{\tau})^2$$

$$\begin{aligned} \partial\text{SSE(R)}/\partial\hat{\mu} = 0 \quad &\Rightarrow \quad \textstyle\sum_i \sum_j (Y_{ij} - \hat{\mu} - \hat{\tau}) = 0 \\ &\Rightarrow \quad \hat{\mu} + \hat{\tau} = \bar{Y}_{..} \end{aligned}$$

Similarly,

$$\begin{aligned} \partial\text{SSE(R)}/\partial\hat{\tau} = 0 \quad &\Rightarrow \quad \textstyle\sum_i \sum_j (Y_{ij} - \hat{\mu} - \hat{\tau}) = 0 \\ &\Rightarrow \quad \hat{\mu} + \hat{\tau} = \bar{Y}_{..} \end{aligned}$$

So, under the reduced model, $\text{SSE(R)} = \sum_i \sum_j (Y_{ij} - \bar{Y}_{\cdot\cdot})^2 = \text{SSTO}$, say

Test statistic for the general linear test:

$$
\begin{aligned}
F^* &= \frac{(\text{SSE(R)}-\text{SSE(F)})}{(\text{df}_R-\text{df}_F)} \Big/ \frac{\text{SSE(F)}}{\text{df}_F} \\
&= \frac{(\text{SSTO}-\text{SSE})}{(n_T-1)-(n_T-r)} \Big/ \frac{\text{SSE}}{(n_T-r)} \\
&= \frac{(\text{SSTR})}{(r-1)} \Big/ \frac{\text{SSE}}{(n_T-r)} \\
&= \frac{\text{MSTR}}{\text{MSE}} \sim F(r-1, n_T-r) \text{ under } H_o
\end{aligned}
$$

So, an α-level test rejects $H_o: \ \tau_1 = \cdots = \tau_r$ if $F^* > F(1-\alpha; r-1, n_T-r)$

Note: SSTO = SSTR + SSE, giving the partition of SSTO considered on pp. 690–691

ANOVA Table 16.3, p. 694
One can derive the expected mean squares, (see pp. 696–698 for the case $n_i = n$)

Example – Kenton Food Company, p. 685
- see SAS example ch16eg1.txt

— break —

Homework: chapter 16, problems 1, 8–10. For problems 8–10, using the given data, just plot the data versus factor level, do the ANOVA (including the F-test), and discuss the results.

Further topics:

SAS ch16eg2.txt: Illustrates relation between regression and ANOVA for quantitative factor

16.8 Regression approach to single factor ANOVA – see ch16eg1.txt

— end of class #2, 1/8/09 —

Distinguish my factor effects model from that in (16.62) on p. 70 with constraints, and discuss estimability

Differences of opinion with authors:

- p. 681, "data analysis usually proceeds in two steps"

- 16.7 (pp. 701–2), constraints for factor effects model

- p. 704: clarify the distinction:

imposing constraints	$\Rightarrow$	interpretation on parameters
no constraints	$\Rightarrow$	just focus on "estimable effects"

16.9 Randomization Tests

An alternative, more fundamental justification than standard model assumptions for testing equality of treatment effects.

What is it? – See Example, p. 713 (illustrative, but too small)

"...both empirical and theoretical studies have shown that the F distribution is a good approximation to the exact randomization distribution when the sample sizes are not small. Thus, randomization alone can justify the F tests as good approximate test, without requiring any assumption of independent, normal error terms." (p. 714)

See Example, p. 714 (of realistic size, though Figure 16.8 is too discrete—way more than necessary)

16.10 Planning of Sample Sizes with Power Approach

The *power* of a test of hypothesis is the probability of rejecting the null hypothesis as a function of the true parameters. In general, a good test will have more power for rejecting the null hypothesis the more false the null hypothesis, the more data one collects, or the better the experiment and method of analysis. For the one-way ANOVA, under our model assumptions, the power of the F test is computed from the *non-central F distribution*, which depend on a single noncentrality parameter ϕ, in addition to the numerator and denominator degrees of freedom.

$$\text{Power} = 1 - \beta = P\{F^* > F(1-\alpha; r-1, n_T - r) \mid \phi\},$$

where $\phi^2 = \frac{1}{\sigma^2}\left[\frac{\sum_i n_i(\tau_i - \bar{\tau}_.)^2}{r}\right]$, for $\tau_. = \sum_i n_i \tau_i / n_T$.

Table B11 (pp. 1337–1341) gives the power as a function of α, ϕ, ν_1 and ν_2, where ν_1 and ν_2 denote numerator and denominator d.f. respectively.

See p. 717 for an example, but this approach is not the most useful for planning experiments.

Use of Table B.12 for Single Factor Studies — a more useful approach.

Table B.12 provides the common sample size n needed, given r, α, β, and Δ/σ, where $\Delta = \max(\tau_i) - \min(\tau_i)$ is the smallest difference in treatment effects one want to be able to detect with the specified power.

Q: How is Δ related to ϕ? A: Based on the worst/hardest case to detect!

Example: (make up numbers)

Homework: chapter 16, problems 26, 27, 29; read 17.1–17.8.