Chapter 9 Ordinary Differential Equations

$$\frac{dy}{dt} = F(t, y), \quad y(t_0) = y_0$$

- Picard's Theorem: If $F(t, y)$ is continuous near (t_0, y_0) ,then there exists a solution $y = y(t)$ for above IVP near $t_{0.}$ This is called local existence theorem. If in addition, $F_y(t, y)$ is also continuous near (t_0, y_0) ,then the solution is unique.
- Section 9.1 Separation of Variables

$$\frac{dy}{dt} = F(t, y) = f(t)\, g(y)$$

General Solution is

$$\int \frac{1}{g(y)} dy = \int f(t)\, d$$

- Example: $y' = t/y$
- Modeling with ODE: y' = rate of change. apply various laws.
- Problem in page 124: Newton's law $mv' = -kv^2$
- Problem in page 125: Newton's Law of cooling $T' = -k(T - T_0)$

- Section 9.2 Mechanics
  - Newton"s method: system internally created force = sum of all external forces exerting on the system (momentum conservation)
  - Hamilton's Method: Energy conservation
- Example (pendulum) page 127
  - Newton's method: $F = ma,\ \ F =$ tangential $gravity = -mg\sin\theta,\ a = L\theta''$ (arclength $= L\theta)$

$$L\theta'' = -mg\sin\theta$$

  - Hamilton's Method:
    * kinetic energy $= mv^2/2 = m\left(L\theta'\right)^2/2$
    * potential energy $=$gravity$\times$vertical distance $= mgL\left(1-\cos\theta\right)$

$$\frac{1}{2}m\left(l\theta'\right)^2 + mgL\left(1-\cos\theta\right) = c$$

  - Hamilton's method $=$ integral of Newton's method
- Example (Mass-spring) page 128
  - Newton's method: $F = ma,\ \ F = -kx$ (Hooks law

$$mx" = -kx$$

- Hamilton's Method: Kinetic energy $= mv^2/2$, potential energy $=$ work down moving m to position x against the spring force

$$\int_0^x ksds = \frac{k}{2}x^2$$

so Hamilton model is

$$\frac{m\left(x'\right)^2}{2} + \frac{k}{2}x^2 = c$$

- Section 9.3 Linear ODEs with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_0 y = u\left(t\right)$$

- $n = 2$: harmonic oscillator
- Linearity: General solution = general solution of homogeneous part + a particular solution
- Differential operator: Set $P\left(t\right) = t^n + a_{n-1}t^{n-1} + ... + a_0$

$$D = \frac{d}{dt}, \quad D^2 = \frac{d^2}{dt^2}, \; ...$$

$$P\left(D\right)y = y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_0 y$$

$$P(D)\,Q(D)\,y = Q(D)\,P(D)\,y$$

$$P(D)\,y = u(t)$$

$$P(D)\,y = u(t)$$

- General solution of homogeneous equations:
  - Polynomial $P(t)$ has total of n real and complex roots including multiplicity. So

$$P(t) = (t - r_1)^{m_1} \dots (t - r_k)^{m_k} \left[(t - \alpha_1)^2 + \beta_1^2\right]^{m_{k+1}} \dots \left[(t - \alpha_l)^2 + \beta_l^2\right]^{m_{k+l}}$$

  - For real root $r_j,\ general$ solution for $(D - r_j)^{m_j}\, y = 0$ is

$$y_j = \left(c_{m_j-1}^{(j)} t^{m_j-1} + c_{m_j-2}^{(j)} t^{m_j-2} + \dots + c_0^{(j)}\right) e^{r_j t}, \quad j = 1, 2, \dots, k$$

  - For each pair of complex root $\alpha_j \pm i\beta_j$, general solution for

$$\left[(D - \alpha_j)^2 + \beta_j^2\right]^{m_{k+j}} y = 0$$

is (for $j = k+1, k+2, ...k+l$)

$$y_j = \left(c^{(j)}_{m_j-1} t^{m_j-1} + c^{(j)}_{m_j-2} t^{m_j-2} + ... + c^{(j)}_0\right) e^{\alpha_j t} \cos \beta_j t$$
$$+ \left(d^{(j)}_{m_j-1} t^{m_j-1} + d^{(j)}_{m_j-2} t^{m_j-2} + ... + d^{(j)}_0\right) e^{\alpha_j t} \sin \beta_j t$$

– So general solution for $P(D)\, y = 0$ is

$$y = y_1 + y_2 + ... + y_k + y_{k+1} + ... + y_{k+l}$$

– Example: $y" + 3y' + 3y = 0$

– Example: $P(t) = (t+1)(t-2)^2 \left[(t+1)^2 + 4\right]^2$. Solve $P(D)\, y = 0$

- General solution of non-homogeneous equations:

$$P(D)\, y = u(t)$$

– Superposition: if y_h is the general solution of homogeneous equations $P(D)\, y = 0$, and if y_p is a particular solution of $P(D)\, y_p = u(t)$, then $y = y_h + y_p$ is the general solution of non-homogeneous equations.

– Particular solution y_p may be found by the method of undetermined coefficients. For instance, if $u(t) = t^n e^{\lambda t} \sin \omega t$, then $y_p = Q(t)\, e^{\lambda t} (A \sin \omega t + B \cos \omega t)$,where $Q(t)$ is a polynomial of degree $n+m$, where m generally depends on whether λ or ωi is root of $Q(t)$

- Section 9.4 Systems of ODEs
  - nth-order ODE is equivalent to a system of first-order of ODE: $y_1 = y, y_2 = y', ..., y_n = y^{(n-1)}$

$$\begin{aligned} y_i' &= y_{i+1}, \quad i = 1, 2, ..., n-1 \\ y_n' &= u(t) - (a_{n-1}y_n + a_{n-2}y_{n-1} + ... + a_0 y_1) \end{aligned}$$

  - Matrix form of first-order linear ODE with constant coefficients: $y" = Ay, \ A = [a_{ij}]$:

$$y_1' = a_{11}y_1 + a_{12}y_2 + ... + a_{1n}y_n$$
$$..,...$$
$$y_n' = a_{n1}y_1 + .a_{n2}y_2 + ... + a_{nn}y_n$$

  - Example in page 136.
  - Solve $2x2$ systems:
    * find all eigenvalues $\det(A - \lambda I) = 0$.
    * for real eigenvalues λ,find eigenvector $\vec{v}$. Then $e^{\lambda t}\vec{v}$ is a solution (sink, source, saddle)
    * for repeated eigenvalue λ, find eigenvector $\vec{v}$, and the second eigenvector $\vec{u} : (A - \lambda I)\vec{u} = \vec{v}$. Then $e^{\lambda t}\vec{v}$ and $te^{\lambda t}\vec{u}$ are two solutions
    * for complex eigenvalue $\lambda = \alpha + \beta i$, find complex eigenvector $\vec{v} = \vec{u}_1 + i\vec{u}_2$. The real part and complex part of $e^{\lambda t}\vec{v}$ are two solutions (spiral sink, source, center)
  - For $n \times n$ system: same except for the case when λ is a repeated eigenvalue of multiplicity m.Then

we need to find kth eigenvector $u_k$by solving

$$\left(A - \lambda I\right)^k u_k = u_{k-1}, \quad k = 1, ..., m, \; u_0 \text{ is an eigenvector}$$

- Method of Exponential of matrix:
  - Recall the Taylor series expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

  - it is convergent for all x.
  - for diagonal matrix $A = diag\left(\lambda_1, ..., \lambda_n\right)$

$$A^k = diag\left(\lambda_1^k, ..., \lambda_n^k\right)$$

  - So as $N \to \infty$

$$\sum_{k=0}^{N} \frac{A^k}{k!} = diag\left(\sum_{k=0}^{N} \frac{\lambda_1^k}{k!}, ..., \sum_{k=0}^{N} \frac{\lambda_n^k}{k!}\right) \to diag\left(e^{\lambda_1}, ..., e^{\lambda_n}\right) = e^A$$

- **Definition of** e^A for general matrix

$$\exp\left(A\right) = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

- Example . Find e^A if

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Sol:

$$e^{At} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

- **Homework** . Find e^A if

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \mu \end{pmatrix}? \qquad \text{(Exercise 1)}$$

- Properties of exponential of matrices:

(a) If $B = T^{-1}AT$, then $e^B = T^{-1}e^A T$

(b) If $AB = BA$, then $\exp(A + B) = e^A e^B$

(c) $\exp(-A) = (\exp(A))^{-1}$

(d) If λ is an eigenvalue of A and V is an associated eigenvector, then e^λ is an eigenvalue of $e^A$and V is an eigenvector of e^A associated with e^λ

(e) $\left(e^{tA}\right)' = Ae^{tA} = e^{tA}A$

- Theorem: Solution of

$$y' = Ay + u(t), \quad y(0) = y_0$$

is

$$y = e^{tA}\left(y_0 + \int_0^t e^{-As} u(s)\, ds\right)$$

In particular, $e^{tA} y_0$ is the solution of homogeneous system $y' = Ay$

- **Frequency-Domain Methods (Chapter 10)**
- Laplace Transform of a function

$$F(s) = L(f)(s) = \int_0^\infty f(t)\, e^{-ts} dt$$

  - $L(e^{at}) = \dfrac{1}{s-a}$
  - $L(\sin at) = \dfrac{a}{s^2 + a^2}$
  - $L(\cos at) = \dfrac{s}{s^2 + a^2}$
- Property: $L(y') = sL(y) - y(0)$

- Solving system: $y = [y_1 \; y_2 \; ...y_n]^T$ be a vector function, A_{nxn} be a matrix

$$y' = Ay$$

Applying Laplace transform, write $Y = L(y)\,.\;then$

$$sY(s) - Y(0) = AY$$
$$Y(s) = (sI - A)^{-1} Y(0)$$

where $R(s) = (sI - A)^{-1}$ is called resolvent. According to Cramer's rule,

$$R(s)\,Y(0) = \frac{1}{\det(sI - A)} adj\,(sI - A) = \sum \frac{1}{(s - \beta_i)^j} \vec{u}_{ij}$$

Therefore,

$$y = \sum L^{-1}\left(\frac{1}{(s - \beta_i)^j}\right) \vec{u}_{ij}$$

- Example in page 152
- Serier solutions: Consider

$$y" + p(t)\,y' + q(t)\,y = f(t)\,,\; y(0) = y_0, y'(0) = y_1$$

Assume that $p(t), q(t), f(t)$ are all analytic functions, i.e., they all have Taylor series representations

$$p(t) = \sum_{k=0}^{\infty} p_k t^k,\ q(t) = \sum_{k=0}^{\infty} q_k t^k,\ f(t) = \sum_{k=0}^{\infty} f_k t^k$$

Look for solution in series form

$$y(t) = \sum_{k=0}^{\infty} y_k t^k$$

Now

$$y' = \sum_{k=0}^{\infty} k y_k t^{k-1} = \sum_{k=0}^{\infty} (k+1) y_{k+1} t^k$$

$$y" = \sum_{k=0}^{\infty} k(k-1) y_k t^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) y_{k+2} t^k$$

Substitute all these serieses into ODE

$$\sum_{k=0}^{\infty} (k+2)(k+1)\, y_{k+2} t^k + \left(\sum_{k=0}^{\infty} p_k t^k\right) \sum_{k=0}^{\infty} (k+1)\, y_{k+1} t^k$$
$$+ \left(\sum_{k=0}^{\infty} q_k t^k\right) \sum_{k=0}^{\infty} y_k t^k = \sum_{k=0}^{\infty} f_k t^k$$

Note that

$$\left(\sum_{k=0}^{\infty} a_k t^k\right) \left(\sum_{k=0}^{\infty} b_k t^k\right) = \left(\sum_{k=0}^{\infty} c_k t^k\right)$$

$$c_k = \sum_{j=0}^{k} a_{k-j} b_j$$

So

$$\left(\sum_{k=0}^{\infty} p_k t^k\right) \sum_{k=0}^{\infty} (k+1)\, y_{k+1} t^k = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} p_{k-j} (j+1)\, y_j\right) t^k$$

$$\left(\sum_{k=0}^{\infty} q_k t^k\right) \sum_{k=0}^{\infty} y_k t^k = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} q_{k-j} y_j\right) t^k.$$

The coefficient of t^k in the left-hand side of the equation is equal to that of the RHS:

$$(k+2)(k+1)\, y_{k+2} + \sum_{j=0}^{k} p_{k-j} (j+1)\, y_j + \sum_{j=0}^{k} q_{k-j} y_j = f_k$$

or

$$y_{k+2} = \frac{1}{(k+2)(k+1)} \left[f_k - \sum_{j=0}^{k} p_{k-j} (j+1)\, y_j - \sum_{j=0}^{k} q_{k-j} y_j \right],$$

$$k = 0, 1, \ldots$$

- Homework (in this note): Exercise 1
- Homework (in textbook): 9.10, 9.15, 9.40, 9.41 (using Exercise 1). 9.42