Chapter 3 Data Acquisition and Manipulation

In this chapter we introduce $z - transform$, or the discrete Laplace Transform, to solve linear recursions.

- Section 3.1 z-transform
  - Given a data stream $x = \{x_0, x_1, x_2, ...\}$, let

$$X = Z(x) = \sum_{k=0}^{\infty} \frac{x_k}{z^k}$$

  - $x \longmapsto X$ is called a $z - transform$
  - Example: $x = \{a, a, a, ...\}\,,\ X = az(z-1)^{-1}$
  - Example: $x = \left\{1, a, a^2, a^3, ...\right\}\,,\ \ X = z(z-a)^{-1}$
  - Data x_i could be a complex number.
  - For instance, the Sine wave $x(t) = \sin \omega t$ is sampled every T seconds to yield the signals

$$x_k = \sin(\omega T k) = \mathrm{Im}\left(e^{k\omega T i}\right),\quad k = 0, 1, 2, ...$$

So by the Euler formula

$$\begin{aligned} X &= \sum_{k=0}^{\infty} \frac{\sin(\omega T k)}{z^k} = \sum_{k=0}^{\infty} \frac{e^{k\omega T i} - e^{-k\omega T i}}{2iz^k} \\ &= \frac{1}{2i} \sum_{k=0}^{\infty} \left(\frac{e^{\omega T i}}{z} \right)^k - \frac{1}{2i} \sum_{k=0}^{\infty} \left(\frac{e^{-\omega T i}}{z} \right)^k \\ &= \frac{1}{2i} \left(1 - \frac{e^{\omega T i}}{z} \right)^{-1} - \frac{1}{2i} \left(1 - \frac{e^{-\omega T i}}{z} \right)^{-1} \\ &= \frac{1}{2i} \frac{z}{z - e^{\omega T i}} - \frac{1}{2i} \frac{z}{z - e^{-\omega T i}} \\ &= \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1} \end{aligned}$$

– Linearity Property:

$$Z(ax + bx) = aZ(x) + bZ(x)$$

– Delay of x : $x_{(-1)} = \{0, x_0, x_1, ...\}$

$$X_{-1} = Z\left(x_{(-1)}\right) = \sum_{k=1}^{\infty} \frac{x_{k-1}}{z^k} = \frac{X}{z}$$

– Delay of $x_{(-1)}$: $x_{(-2)} = \{0, 0, x_0, x_1, ...\} = \left(x_{(-1)}\right)_{(-1)}$

$$X_{-2} = Z\left(x_{(-2)}\right) = \frac{1}{z} Z\left(x_{(-1)}\right) = \frac{X}{z^2}$$

– In general

$$Z\left(x_{(-j)}\right) = \frac{X}{z^j} \tag{1}$$

– Forward of x : $x_{(+1)} = \{x_1, x_2, ...\}$

$$X_{+1} = Z\left(x_{(+1)}\right) = \sum_{k=0}^{\infty} \frac{x_{k+1}}{z^k} = z \sum_{k=0}^{\infty} \frac{x_{k+1}}{z^{k+1}} = z \sum_{k=0}^{\infty} \frac{x_k}{z^k} - x_0 z = z\left(X - x_0\right) = zX - x_0 z$$

In general for kth forward of $x : x_{(+k)}$, its $Z - transform$ $X_{+k} = Z\left(x_{(+k)}\right)$

$$X_{+k} = z^k X - x_0 z^k - x_1 z^{k-1} - x_2 z^{k-2} - ... - x_{k-1} z$$

– For any sequence $x = \{x_k\}_{k=0}$, for convenience we may write

$$Z\left(x_k\right) = Z\left(x\right).$$

– For a shift $\{x_{k-1}\}_{k=0} = \{x_{-1}, x_0, x_1, x_2, ...\}$ of x with given initial value x_{-1},we write

$$\begin{aligned} X_{k-1} &= Z\left(x_{k-1}\right) = Z\left(\{x_{k-1}\}_{k=0}\right) \\ &= x_{-1} + \frac{x_0}{z} + \frac{x_1}{z^2} + ... \\ &= x_{-1} + \frac{1}{z}\left(x_0 + \frac{x_1}{z} + ...\right) \\ &= x_{-1} + \frac{Z\left(x\right)}{z} \end{aligned}$$

– For a pth shift $\{x_{k-p}\}_{k=0} = \{x_{-p}, x_{-p+1}, ...x_{-1}, x_0, x_1, x_2, ...\}$with given initial values $x_{-p}, x_{-p+1}, ...x_{-1}$,we have accordingly

$$\begin{aligned} X_{k-p} &= Z\left(x_{k-p}\right) = Z\left(\{x_{k-p}\}_{k=0}\right) \\ &= x_{-p} + \frac{x_{-p+1}}{z} + ... + \frac{x_{-1}}{z^{p-1}} + \frac{x_0}{z^p} + \frac{x_1}{z^{p+1}} + ... \\ &= x_{-p} + \frac{x_{-p+1}}{z} + ... + \frac{x_{-1}}{z^{p-1}} + \frac{1}{z^p}\left(x_0 + \frac{x_1}{z} + ...\right) \\ &= x_{-p} + \frac{x_{-p+1}}{z} + ... + \frac{x_{-1}}{z^{p-1}} + \frac{1}{z^p} Z\left(x\right) \end{aligned} \quad (2)$$

– For instance, If $y_k = ax_{k-1} + bx_{k-2}$, and

$$\begin{aligned} Y &= Z\left(y_k\right) = Z\left(ax_{k-1} + bx_{k-2}\right) = aZ\left(x_{k-1}\right) + bZ\left(x_{k-2}\right) \\ &= a\left(x_{-1} + \frac{Z\left(x_k\right)}{z}\right) + b\left(x_{-2} + \frac{x_{-1}}{z} + \frac{Z\left(x_k\right)}{z^2}\right) \\ &= ax_{-1} + bx_{-2} + \frac{bx_{-1}}{z} + \left(\frac{a}{z} + \frac{b}{z^2}\right) Z\left(x\right) \end{aligned} \tag{3}$$

- Section 3.2 Linear Recursions

– Consider the equation for all $k = n, n+1, ...$

$$x_k = a_1 x_{k-1} + a_2 x_{k-2} + ... + a_n x_{k-n}$$

where a_k are fixed constant. $x_{-1}, x_{-2}, ..., x_{-n}$ are given initially, and are called initial data.

– We can use $z - transform$ to solve x by using the above calculation:

(i) Apply $z - transform$ to both sides.

(ii) note that $Z\left(x_{k-p}\right)$ is given by (2)

(iii) Solve $X = Z\left(x\right)$

(iv) Recover x_k

– Example (page 34) Solve

$$x_k = x_{k-1} + 2x_{k-2}, \quad x_{-1} = 1/2, \quad x_{-2} = -1/4$$

Solution: By (3) with $a = 1, b = 2$, we have

$$\begin{aligned} Z(x_k) &= ax_{-1} + bx_{-2} + \frac{bx_{-1}}{z} + \left(\frac{a}{z} + \frac{b}{z^2}\right) Z(x_k) \\ &= \frac{1}{z} + \left(\frac{1}{z} + \frac{2}{z^2}\right) Z(x_k) \end{aligned}$$

Set $w = 1/z$, then

$$\begin{aligned} Z(x_k) &= \frac{1}{z}\left(1 - \frac{1}{z} - \frac{2}{z^2}\right)^{-1} = \frac{w}{1 - w - 2w^2} = \frac{w}{(1-2w)(1+w)} \\ &= \frac{1}{3(1-2w)} - \frac{1}{3(1+w)} \\ &= \frac{1}{3}\sum_{k=0} (2w)^k - \frac{1}{3}\sum_{k=0} (-w)^k = \frac{1}{3}\sum_{k=0} \left[2^k - (-1)^k\right] w^k \\ &= \sum_{k=0} \frac{2^k - (-1)^k}{3} \frac{1}{z^k} \end{aligned}$$

Ans:

$$x_k = \frac{2^k - (-1)^k}{3} \text{ for } k > 0$$

– Problem in page 35: $x_k = ax_{k-1} + b$, for $k = 1, 2, ...,\ x_0 = 0.7\%$. Find x_k. There are two ways to solve it.

  ∗ (1) Set $y_k = x_{k+1}$. So $y_{-1} = x_0 = 0.7\%$ is given, and $y_k = ay_{k-1} + b$ for $k = 0, 1, ...$ Use the relation we derived earlier,

$$Y = aY_{-1} + Z(b) = a\left(z^{-1}Y + y_{-1}\right) + \frac{bz}{z-1}$$

$$\left(1 - az^{-1}\right)Y = ay_{-1} + \frac{bz}{z-1}$$

$$\begin{aligned} Y &= \frac{ay_{-1}}{1 - az^{-1}} + \frac{bz}{(z-1)\left(1 - az^{-1}\right)} = \frac{(b + ay_{-1})\,z^2 - ay_{-1}z}{(z-1)(z-a)} \\ &= (b + ax_0) + \frac{A}{(z-1)} + \frac{B}{(z-a)} \end{aligned}$$

  ∗ (2) From $x_0 = ax_{-1} + b$, we solve $x_{-1} = (x_0 - b)/z$. Then, we proceed to solve

$$\begin{aligned} X &= \frac{(b + ax_{-1})\,z^2 - ax_{-1}z}{(z-1)(z-a)} = \frac{x_0z^2 - ax_{-1}z}{(z-1)(z-a)} \\ &= x_0 + \frac{C}{(z-1)} + \frac{D}{(z-a)} \end{aligned}$$

– Convolution: For $x = \{x_k\}, h = \{h_k\}$, the discrete convolution product $y = h * x$ is defined as follows

$$y_k = h_0 x_k + h_1 x_{k-1} + ... + h_k u_0 = \sum_{j=0}^{k} h_j x_{k-j} = \sum_{j=0}^{k} x_j h_{k-j}$$

– In the frequency domain, $Z(h * u) = Z(h) Z(u)$, or $Y = HU$. This is because the above expression indicate power series product. In fact

$$\left(\sum_{k=0}^{\infty} a_k z^k\right)\left(\sum_{k=0}^{\infty} b_k z^k\right)$$
$$= \left(a_0 + a_1 z + a_2 z^2 + ...\right)\left(b_0 + b_1 z + b_2 z^2 + ...\right)$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + ...$$

– Examples: (1) for $\delta^{(0)} = \{1, 0, 0, 0, ...\}$, $u * \delta^{(0)} = u$, (2) $\delta^{(1)} = \{0, 1, 0, 0, 0, ...\} = \delta_{(-1)}$, $u * \delta^{(1)} = u_{(-1)}$, (3) $u * \delta^{(j)} = u_{(-j)}$

- Section 3.3 Filters

– A filter F is a device or algorithm that turns one stream of signals to another more useful one. We assume following three properties

(i) F is linear: $F(au + bv) = aF(u) + bF(v)$

(ii) F is causal, i.e., the output depends on past and current inputs but not future inputs. In other words, if $y = F(u)$, then

$$u_k = 0 \text{ for all } k < k_0 \text{ implies } y_k = 0 \text{ for all } k < k_0.$$

(iii) F is time invariant: if $y = F(u)$, then the same is true for delay shift, i.e.,

$$F\left\{u_{(-k_0)}\right\} = \left\{y_{(-k_0)}\right\}$$

In other words, if $F : u \to y$, then

$$F : \{u_k, u_{k+1}, ...\} \to \{y_k, y_{k+1}, ...\}$$

– Let $\delta = \{1, 0, 0, 0, ...\}$ be the unit impulse, and $h = F(\delta) = \{h_0, h_1, ...\}$ .Then for any u

$$y = F(u) = h * u$$

where $*$ stands for discrete convolution product defined as follows

$$y_k = h_0 u_k + h_1 u_{k-1} + ... + h_k u_0 = \sum_{j=0}^{k} h_j u_{k-j} = \sum_{j=0}^{k} u_j h_{k-j}$$

– Proof: Let $\delta^{(j)} = \delta_{(-j)} = \{0, ..., 0, 1, 0, ...\}$ be a $j - delay$, where 1 is in $j - th$ position. Then by property #3, $F\left(\delta^{(j)}\right)=h_{(-j)}=\{0,...,0,h_0,h_1,...\}$Now

$$u=\sum_{j=0}^{\infty}u_j\delta_{(-j)}$$

$$y=F\left(u\right)=\sum_{j=0}^{\infty}u_jF\left(\delta_{(-j)}\right)=\sum_{j=0}^{\infty}u_jh_{(-j)}$$

. Note that

$$\left(h_{(-j)}\right)_k=0\ \text{ if } k<j,\ \ \left(h_{(-j)}\right)_k=h_{k-j}\ \text{ if } k\geq j$$

so for any k

$$y_k=\sum_{j=0}^{\infty}u_j\left(h_{(-j)}\right)_k=\sum_{j=0}^{\infty}u_jh_{k-j}=\sum_{j=0}^{k}u_jh_{k-j}=(h*u)_k$$

– $H\left(z\right)=Z\left(h\right)$ is called transfer function of F with impulse response h

– Examples: (1) $u*\delta^{(j)}=u_{(-j)}$, (2) $Z\left(u_{(-j)}\right)=Z\left(u*\delta^{(j)}\right)=UZ\left(\delta^{(j)}\right)=Uz^{-j}$

– Example (page 38) Consider the filter with the finite impulse response (FIR)

$$h = \{1/2, 1/2, 0, 0, ...\} = \frac{1}{2}\delta^{(0)} + \frac{1}{2}\delta^{(1)}$$

For any signal $u = \{u_0, u_1, ...\}$,

$$y = u * h = \frac{1}{2}u * \delta^{(0)} + \frac{1}{2}u * \delta^{(1)} = \frac{1}{2}u + \frac{1}{2}h_{(-1)}.$$

This can also be seen directly

$$y_k = \sum_{j=0}^{k} u_j h_{k-j} = u_k h_0 + u_{k-1} h_1 = \frac{u_k + u_{k-1}}{2}, \quad y_0 = u_0 h_0 = \frac{u_0}{2}$$

So

$$y = \frac{u + u_{(-1)}}{2}, \quad Y = \frac{1}{2}U + \frac{1}{2z}U = \left(\frac{1 + z^{-1}}{2}\right) U$$

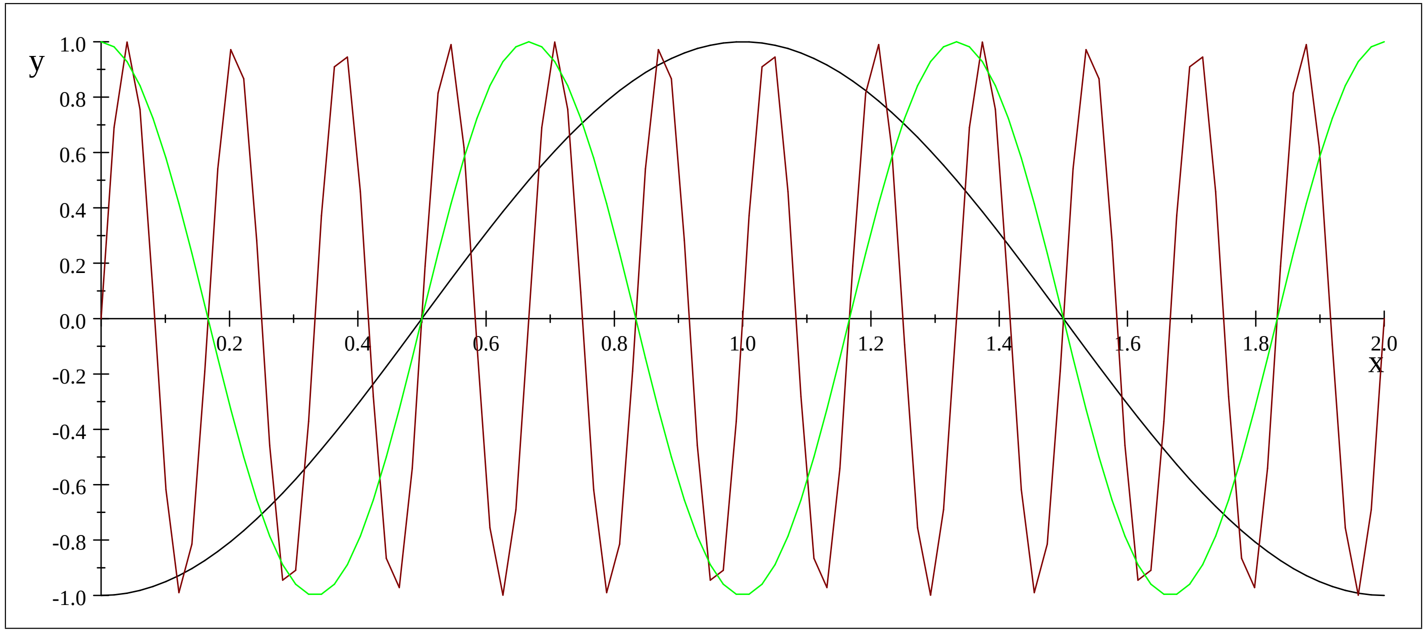
The transfer function is

$$H = \frac{1 + z^{-1}}{2}$$

– Problem (page 38): Suppose that in a room with various sounds. One want to eliminate frequency 60 Hz sound wave. In other words, the component $A \sin(120\pi t)$ needs to be eliminated. In theory, to do so, we need to add

$$-A\sin(120\pi t) = A\sin(120\pi t - \pi) = A\sin\left(120\pi(t - \frac{1}{120}\right)$$

The black curve is sine wave of frquency $60\ Hz$ with period $1/60$ the red has frequency $720\ Hz$ with period $1/720$. If we sample using $720\ Hz$, i.e., pick up a signal every $T = 1/720$ second, it will meet the max of the black wave at $t = 1/120 = 6T,\ 3/120 = 18T,\ 5/120 = 30T, 42T, ...,$ i.e., every $12T$.



$$scale = 1 : \frac{1}{120} = 6T$$

Let $u_k = signal$ picked up at time kT. So $u_6, u_{18}, u_{30,}...$ will be $60\ Hz$ signal. To cancel it, we use the min point: at the second max point u_{18}, we use the previous min point u_{12} to cancel it. Thus the filter $y_k = u_k + u_{k-6}$, or $Y = \left(1 + z^{-6}\right) U$

- – Downside: Green wave has frequency $180\ Hz$. This filter could also cancel this frequency sound.
- Section 3.4 Stability
  - We say a filter with impulse response h is stable if bounded inputs u yield bounded outputs y.
  - Theorem: A filter is stable iff the transfer function $H(z)$ absolutely converges on the unit circle $|z| = 1$, i.e.,

$$\sum_{j=0}^{\infty} |h_j| < \infty$$

  - If H has a pole at $z = z_0$ of order k, i.e.,

$$H(z) = \frac{G(z)}{(z - z_0)^k}, \quad G(z) \text{ is bounded near } z_0$$

then it is stable only if $|z_0| < 1$. This is because for all $|z| > 1$,

$$|z - z_0| \geq |z| - |z_0| > 1 - |z_0| > 0.$$

Thus

$$|H(z)| = \frac{|G(z)|}{|z - z_0|^k} \leq \frac{|G(z)|}{(1 - |z_0|)^k} \text{ is bounded for all } z.$$

– Example in page 40.

- Section 3.5 Polar and Bode Plots
- We shall introduce two graphing methods to exam efficacy of filters
- Theorem. Let F be a stable filter with real impulse response $h = \{h_0, h_1, ...\}$ and transfer function $H(z) = Z(h)$.Then after transients have died away, the response to the sinusoidal signal $u_k = \sin \omega kT$ is also a sinusoid $y_k = r \sin(\omega kT + \phi)$ of the same frequency but different amplitude and phase angle

$$r = \left| H\left(e^{i\omega T}\right) \right|, \quad \phi = \arg H\left(e^{i\omega T}\right).$$

- Proof: Recall that $y = F(u) = h * u$. In particular

$$y_k = \sum_{j=0}^{k} h_j u_{k-j}$$

For $u_k = \xi^k$,

$$y_k = \sum_{j=0}^{k} h_j \xi^{k-j} = \xi^k \sum_{j=0}^{k} h_j \xi^{-j}$$
$$= \xi^k \sum_{j=0}^{\infty} h_j \xi^{-j} - \xi^k \sum_{j=k+1}^{\infty} h_j \xi^{-j} = \xi^k H(\xi) + o(1),$$

where $o(1) \to 0$ as $k \to \infty$ (by assumption of after dying away)

Set

$$H(\xi) = re^{i\phi}, \quad \xi = e^{i\omega T}$$

then, the response to $u_k = \xi^k = e^{i\omega kT}$ is

$$y_k = \xi^k H(\xi) + o(1) = re^{i\phi}\xi^k + o(1) = re^{i(\omega kT + \phi)} + o(1)$$
$$= r[\cos(\omega kT + \phi) + i \sin(\omega kT + \phi)] + o(1)$$

Since h_k are real, the response to the imaginary parts of $e^{i\omega kT}$, i.e., $\mathrm{Im}(u_k) = \sin \omega kT$, should be the imaginary parts of the response to u_k

$$\mathrm{Im}(y_k) == r \sin(\omega kT + \phi) + o(1).$$

- The graph $(r(\omega), \phi(\omega))$ in polar coordinate is called Polar plots. $T = smapling$ period, $1/T$ sampling rate.

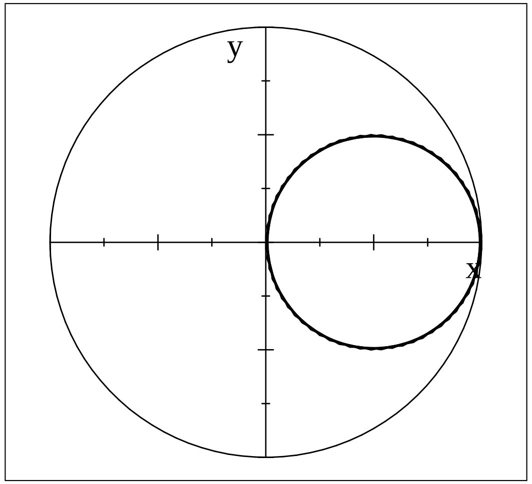
- Two separate graphs, $r = r(\omega)$, and $\phi = \phi(\omega)$ are called $Bode$ plots (Bode-ee).
- Bode plot for r is often scaled in decibels (dB) of powers $20 \log_{10} r$ (see section 3.8)
- Example in page 41: $h = \{1/2, 1/2, 0, 0, 0, ...\}$, $y_k = (u_k + u_{k-1})/2$, $H(z) = \left(1 + z^{-1}\right) 2$. So

$$H\left(e^{i\omega T}\right) = \frac{1 + e^{-i\omega T}}{2} = \frac{1 + \cos \omega T - i \sin \omega T}{2}$$

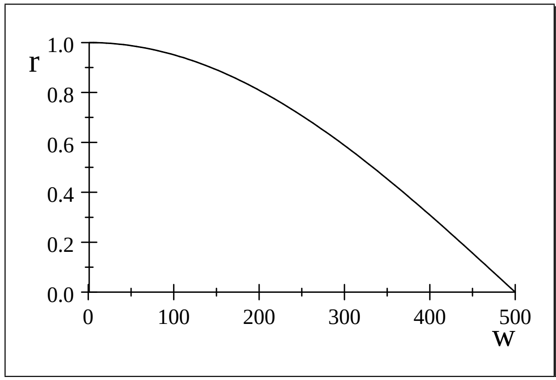
So

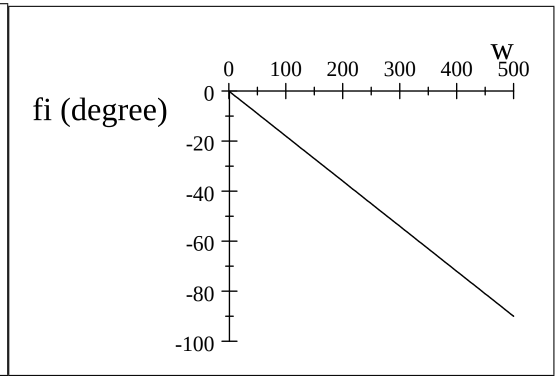
$$r = \sqrt{\frac{1 + \cos \omega T}{2}}, \quad \phi = -\arctan \frac{\sin \omega T}{1 + \cos \omega T}$$

– Polar plot



– Bode plots





- Section 3.7 Closing the loop

– See chart in page 47. We add a feedback filter H to improve stability

$$(U - HY)\,P = Y \implies Y = \frac{P}{1 + HP}U$$

– For any plant

$$P = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

that has a pole at $z = a$. So for $a > 1$, it is unstable. If we add a constant filter $H = k$. Then

$$Y = \frac{P}{1 + kP}U = \frac{z}{(1 + k)\,z - a}U$$

that has a pole at

$$z = \frac{a}{1 + k} < 1 \quad \text{if } k > a - 1$$

– Example (at bottom of page 48): For the planr $P = \left(1 - 2z^{-1}\right)^{-1}, \ \ a = 2$. We add filter $k = 3/2 > 1$ will make it stable.

- Homework: 3.1, 3.2, 3.7, 3.8, 3.18, 3.20, 3.21
- Find explicit formula for Fibonacci sequence: $F_0 = F_1 = 1, \ F_{k+2} = F_k + F_{k+1}$ for $k = 0, 1, 2, ...$ Then

write a Matlab routine to varify your answer. Would answer be different if we start with

$$F_0 = 1, F_1 = 2, F_{k+2} = F_k + F_{k+1} \text{ for } k = 0, 1, 2, ...?$$

Or

$$F_0 = 2, F_1 = 3, F_{k+2} = F_k + F_{k+1} \text{ for } k = 0, 1, 2, ...?$$

- Project: 3.10 (optional)