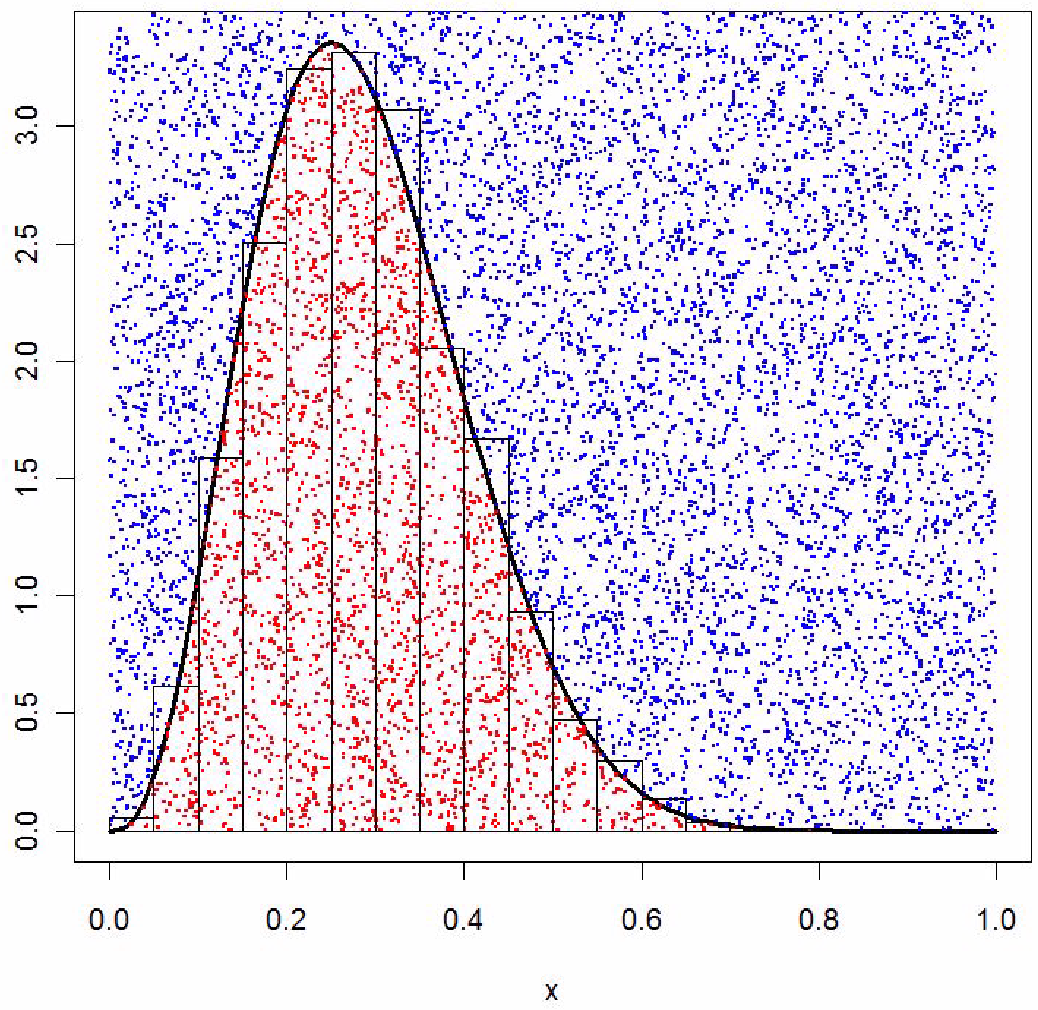
Chapter 2 The Monte Carlo Method

The Monte Carlo Method stands for a broad class of computational algorithms that rely on random samplings. It is often used in physical and mathematical problems and is most useful when it is difficult or impossible to use other algorithms. There are mainly three distinct classes of applications: numerical integration, optimization, and generating draws from a probability distribution. (https://cse.sc.edu/~terejanu/files/tutorialMC.pdf)

- Section 2.1 Computing Integrals
  - Simple Monte Carlo for finding min and max of $y = f(x)$
    - randomly generate points $x_1, x_2, ...$
    - $Min = Max = f(x_1)$,
    - if $f(x_2) < f(x_1)$, then replace $Min = f(x_2)$, $Max = f(x_1)$, $\quad continues$
  - Consider the area below the curve $y = f(x)$ $inside$ the rectangle of area R. We randomly generate a point (x_1, y_1) inside the rectangle. If $y_1 \leq f(x_1)$, then this point is below the curve. If in N trials, M points are below the curve, then:

$$area\ of\ under\ the\ curve\ \approx \frac{M}{N} R$$


0.0
0.5
1.0
1.5
2.0
2.5
3.0
0.0
0.2
0.4
0.6
0.8
1.0
x

– In theory, as $N \to \infty$, the limit is exactly the area.

– Monte Carlo Method for integrals:

  * Recall that $I = \int_a^b f(x)\,dx$ is the signed area between the curve $y = f(x)$ and $x - axis$. If $f(x) \geq 0$ it is the area.

  * We may use the above method to approximate the integral as follows:

    · Set $c = \min\{f(x) : a \leq x \leq b\}$, $d = \max\{f(x) : a \leq x \leq b\}$

    · update: $c = \min\{c, 0\}$, $d = \max\{d, 0\}$

    · N is total trials, initialize $N = M = I = 0$

    · $Each$ trial, randomly generate $a \leq x_1 \leq b, c \leq y_1 \leq d$

    · If $0 \leq y_1 \leq f(x_1)$, then $M = M + 1$

    · If $0 \geq y_1 \geq f(x_1)$, then $M = M - 1$

    · The above two step can be combined as: If $(f(x_1) - y_1)\,y_1 \geq 0$, then $M = M + sign(y_1)$

    · $I = (M/N)(b - a)(d - c)$

  * For double integrals, we use volumes instead.

– Read Example in page 22.

– Exercise 1: Use Monte Carlo to find the following integral within an error of 10^{-5}.What is N?

$$\int_0^2 e^{x^2} \cos x dx$$

- Experiment: try using normal distribution insteat of uniform distribution, compare results.
- Monte Carlo is not very efficient. It is used only for very complicated situations.

- Section 2.2 Mean Time between Failures
- Consider a system has several components. Each has a failure time that is normally distributed with mean μ_i and variance σ_i^2. Then the system failure time is

$$T = \min\{T_1, T_2, ..., T_n\}$$

- Analytic solution for $E[T]$: Let F_i be the CDF of T_i. Note that $T(\omega) \leq T_i(\omega)$,and

$$T(\omega) > t \iff T_i(\omega) > t \text{ for } i = 1, 2, ..., n$$

$$\{\omega : T(\omega) > t\} = \cap_{i=1}^{n} \{\omega : T_i(\omega) > t\}$$

Then CDF F of T is (assuming indenpendence)

$$\begin{aligned} F(t) &= P(T \le t) = 1 - P(T > t) \\ &= 1 - P\left(\cap_{i=1}^{n} \{\omega : T_i(\omega) > t\}\right) \\ &= 1 - \prod_{i=1}^{n} P(\omega : T_i(\omega) > t) \\ &= 1 - \prod_{i=1}^{n} (1 - F_i(t)) \end{aligned}$$

and

$$dF(t) = -d\left(\prod_{i=1}^{n}(1 - F_i(t))\right) = \sum_{k=1}^{n}\left(\prod_{i=\neq k}^{n}(1 - F_i(t))\right) dF_i$$

$$E[T] = \sum_{k=1}^{n}\int_{-\infty}^{\infty} t\left(\prod_{i=\neq k}^{n}(1 - F_i(t))\right) dF_i(t)$$

– This integral is so difficult.

– Monte Carlo for mean and variance:

$$\mu = E[X] = \lim_{N\to\infty} \frac{X(\omega_1) + X(\omega_2) + ... + X(\omega_N)}{N} = \lim_{N\to\infty} \left(\frac{X(\omega_1)}{N} + \frac{X(\omega_2)}{N} + ... + \frac{X(\omega_N)}{N} \right)$$

$$\sigma^2 = E\left[(X-\mu)^2\right] = E\left[X^2 - 2\mu X - \mu^2\right] = E\left[X^2\right] - 2\mu E[X] + \mu^2 = E\left[X^2\right] - \mu^2$$

– Exercise #2: MC for $E[T]$ (See Routine in page 23)

- Section 2.3 Servicing Requests
- Monte Carlo for random variables, means/variances
  - Lemma Suppose that CDF $F_X(x)$ of a random variable X is continuous and strictly increasing. Then $F_X(X) = Y\ \tilde{}U(0,1)$, $i.e.$, Y is uniform distribution in $[0,1]$.
    * Proof: Using graph of $y = F_X(x)$. For any $0 < y_0 < 1$, there is a unique x_0 such that $y_0 = F_X(x_0)$ .Since F_X is strictly increasing,

$$\{x : F_X(x) \le y_0\} = (-\infty, x_0]$$

By definition, $y_0 = P(X \le x_0)$ .From the above, we see that if

$$\omega \in \{\omega : F_X(X(\omega)) \le y_0\}$$

then

$$X(\omega) \in \{x : F_X(x) \le y_0\} = (-\infty, x_0]$$
$$X(\omega) \le x_0$$

So

$$\omega \in \{\omega : X(\omega) \le x_0\}.$$

Consequently

$$\{\omega : F_X(X(\omega)) \le y_0\} \subset \{\omega : X(\omega) \le x_0\}.$$

Since F_X is increasing, we actually have

$$\{\omega : F_X(X(\omega)) \le y_0\} = \{\omega : X(\omega) \le x_0\}.$$

So

$$F_Y(y_0) = P(F_X(X) \le y_0) = P(X \le x_0) = y_0$$

– Now, for any given $F(x)$, we first randomly generate a sample $y\tilde{\ }U(0,1)$, then calculate $x = F^{-1}(y)$ as a sample for F.

– Estimate $E[g(X)]$ for given CDF $F_X(x)$ or PDF ρ_X :
  $*$ For any n, generate a sequence of random variables x_i as above with CDF F_X

* compute

$$\frac{1}{n}\sum_{i=1}^{n} g\left(x_i\right) \longrightarrow E\left[g\left(X\right)\right] = \int g\left(x\right) dF_X = \int_{-\infty}^{\infty} g\left(x\right) \rho_X\left(x\right) dx$$

* In particular,

$$\frac{1}{n}\sum_{i=1}^{n} x_i \longrightarrow E\left[X\right]$$

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \longrightarrow E\left[X^2\right] - E\left[X\right]^2 = \sigma^2$$

* Central Limit Theory: The average of a large number independent random variable converges to a normal distribution. More precisely, let X_i be a random variable with mean μ and variance σ^2, then

$$\frac{\sum_{i=1}^{n}\left(X_i - \mu\right)}{\sigma\sqrt{n}} \longrightarrow N\left(0,1\right), \quad \text{as } n \longrightarrow \infty$$

* Error is $O\left(1/\sqrt{n}\right)$

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} g\left(X_i\right) - E\left[g\left(X\right)\right] &= \frac{\sum_{i=1}^{n}\left(g\left(X_i\right) - E\left[g\left(X\right)\right]\right)}{n} \\
&= \frac{\sum_{i=1}^{n}\left(g\left(X_i\right) - E\left[g\left(X\right)\right]\right)}{\sigma\sqrt{n}}\frac{\sigma}{\sqrt{n}} \\
&\longrightarrow N\left(0,1\right)\frac{\sigma}{\sqrt{n}}
\end{aligned}$$

- Consider checkout time in a library or grocery story. Assume the time T between requests for services has the exponential distribution

$$\rho\left(t\right) = \frac{e^{-t/a}}{a}, \quad \text{for } t > 0$$

  - Assume there are m checkout lines. If the first one is busy, then the requested is handed out to the second, and so on. If all lines are busy, the request is rejected.
  - Assume that each line processes each request in time S that is normally distributed with mean μ and variance σ^2.

– The CDF for T is

$$y = F(t) = \int_0^t \frac{e^{-x/a}}{a} dx = 1 - e^{-t/a}$$

– The inverse function F^{-1} is

$$t = -a \ln(1 - y)$$

– So there exists a uniform distribution in $[0, 1]$, Y ~ $U[0, 1]$

$$T = -a \ln(1 - Y)$$

– See routine in page 25

- Homework:
  – Exercise 1: Using MC to find the integral (see Example in page 22), within an error of 10^{-5}.What is the smallest N?

  – Exercise 2: Run MC Routine in page 23. Then write your own routine for the following cases:
  (a) There are five devices $T_1, ..., T_5$, which have normal distributions with mean 11,12,13,14,15, and standard deviation $1, 2, 3, 4, 5$, respectively.
  (b) Repeat (a) with uniform distribution instead (with the same neans and standard deviations). (Hint: $U\left[\mu - \sigma\sqrt{3}, \mu + \sigma\sqrt{3}\right]$ has mean μ and variance σ^2)

– Exercise 3: Write a MC routine to compute $E[X]$ and $\sigma^2[X]$ with given CDF of X. Then test your MC routine using exponential distribution with $\lambda = 2$ and $\lambda = 5$.

– Exercise 4: Write a Matlab routine to generate $N(0,1)$ (e.g., find mean and variance) using using "rand" command only (you cannot use "randn" command) and Lemmas of page 25. Test your results in 2(a).

– ■Optional■Develop a Matlab routine for MC integration of general integral $\iint_D f(x,y)\,dxdy$ for any function $f(x,y)$ in and domain D. Note that D is not necessarily a rectanglar domain. For instance, D could be the unit disk. Then, (accurate up to 10^{-3})
(i) use your program to compute

$$\int_0^2 dy \int_{-1}^2 e^{-(x^2 y)} \sin x \cos y dx$$

(ii) Let B_1 be the unit disk, i.e., $B_1 : x^2 + y^2 \leq 1$, use MC to find

$$\iint\limits_{B_1} e^{-(x^2 y)} \sin x \cos y dxdy$$