Chapter 12 Divided Differences

- Section 12.1 Euler's Method
- finite difference approximation of derivative: given time step $0 < h << 1$,

$$\frac{dy(t)}{dt} = \lim_{\delta \to 0} \frac{y(t+\delta) - y(t)}{\delta} \simeq \frac{y(t+h) - y(t)}{h}$$

- ODE $y' = f(t, y)$ can be discretized as

$$\frac{y(t+h) - y(t)}{h} = f(t, y(t))$$

or

$$y(t+h) = y(t) + hf(t, y(t))$$

For IVP

$$y' = f(t, y), \; y(t_0) = y_0$$

we may discretize using time step h :

$$\begin{aligned} t_1 &= t_0 + h \\ t_{k+1} &= t_k + h = t_0 + (k+1)h, \quad k = 1, 2, 3, ... \end{aligned}$$

$$y(t_{k+1}) = y(t_k) + hf(t_k, y(t_k))$$

Let $y_k = y(t_k)$, we have Euler's scheme

$$\begin{aligned} t_{k+1} &= t_k + h = t_0 + (k+1)h, \\ y_{k+1} &= y_k + hf(t_k, y_k) \end{aligned}$$

- We can thus obtain a discrete solution: (t_k, y_k). We then use "Spline" to find approximation solution
- Section 12.2 Systems

$$\begin{aligned} x' &= f(t, x, y), \ x(t_0) = x_0 \\ y' &= g(t, x, y), \ y(t_0) = y_0 \end{aligned}$$

In vector form

$$X' = F(t, X), \ X = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \ F(t, X) = \begin{bmatrix} f(t, x, y) \\ g(t, x, y) \end{bmatrix}$$

Euler's scheme:

$$\begin{aligned} t_{k+1} &= t_k + h = t_0 + (k+1)h, \\ X_{k+1} &= X_k + hF(t_k, X_k) \end{aligned}$$

- Section 12.3 PDEs

- Consider the heat equation in $[0,1]$ with the Dirichlet bdy:

$$u_t = u_{xx}(t,x), \quad t>0,\ 0<x<1$$
$$u(0,x) = f(x),\ u(t,t) = a(t),\ u(t,1) = b(t)$$

- Time step h.Interval $[0,1]$ is divided into n subintervals with length $\delta = 1/n$

$$t_k = kh,\ k = 0,1,2,...$$
$$x_i = i\delta = \frac{i}{n},\ i = 0,1,...,n$$
$$u(t_k, x_i) = u_i^k$$

- Central difference for u_{xx} :

  – forward for u_x:

$$u_x(t_k,x_i) = \lim_{d\to 0} \frac{u(t_k, x_i+d) - u(t_k,x_i)}{d} \simeq \frac{u(t_k,x_i+\delta) - u(t_k,x_i)}{\delta} = \frac{u_{i+1}^k - u_i^k}{\delta}$$

$$u_x(t_k,x_{i-1}) = \lim_{d\to 0} \frac{u(t_k, x_{i-1}+d) - u(t_k,x_{i-1})}{d} \simeq \frac{u(t_k,x_{i-1}+\delta) - u(t_k,x_{i-1})}{\delta} = \frac{u_i^k - u_{i-1}^k}{\delta}$$

  – This only works for $i = 0,1,2,...,n-1$.

– Euler's is forward:

$$u_t(t_k, x_i) = \frac{u(t_{k+1}, x_i) - u(t_k, x_i)}{h} = \frac{u_i^{k+1} - u_i^k}{h}$$

– backward for u_{xx} :

$$\begin{aligned} u_{xx}(t_k, x_i) &= \lim_{d \to 0} \frac{u_x(t_k, x_i + d) - u_x(t_k, x_i)}{d} \simeq \frac{u_x(t_k, x_i - \delta) - u_x(t_k, x_i)}{-\delta} \\ &= \frac{u_x(t_k, x_i) - u_x(t_k, x_{i-1})}{\delta} \\ &= \frac{\dfrac{u_{i+1}^k - u_i^k}{\delta} - \dfrac{u_i^k - u_{i-1}^k}{\delta}}{\delta} = \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{\delta^2} \end{aligned}$$

– This only works for interior points $x_i, \quad 0 < i < 1$

– at $i = 0$

$$u_0^k = u(t_k, x_0) = u(t_k, 0) = a(t_k) = a^k$$

– at $i = n$,

$$u_n^k = u(t_k, x_n) = u(t_k, 1) = b(t_k) = b^k$$

- Difference equation for $u_t = u_{xx}$

$$u_t\left(t_k, x_i\right) = u_{xx}\left(t_k, x_i\right),$$

$$\frac{u_i^{k+1} - u_i^k}{h} = \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{\delta^2}$$

$$\begin{aligned} u_i^{k+1} &= u_i^k + \lambda\left(u_{i+1}^k - 2u_i^k + u_{i-1}^k\right) \\ &= \lambda u_{i-1}^k + (1-2\lambda)\, u_i^k + \lambda u_{i+1}^k \end{aligned}$$

  – $\lambda = hl\delta^2 = hn^2$.
  – In matrix form

$$\begin{bmatrix} u_1^{k+1} \\ u_2^{k+1} \\ u_3^{k+1} \\ \vdots \\ u_{n-2}^{k+1} \\ u_{n-1}^{k+1} \end{bmatrix} = \begin{bmatrix} 1-2\lambda & \lambda & \cdots & \cdots & \cdots & 0 \\ \lambda & 1-2\lambda & \lambda & \cdots & 0 & 0 \\ 0 & \lambda & 1-2\lambda & \lambda & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1-2\lambda & \lambda \\ 0 & 0 & 0 & \cdots & \lambda & 1-2\lambda \end{bmatrix} \begin{bmatrix} u_1^k \\ u_2^k \\ u_3^k \\ \vdots \\ u_{n-2}^k \\ u_{n-1}^k \end{bmatrix} + \begin{bmatrix} \lambda a^k \\ 0 \\ 0 \\ \vdots \\ 0 \\ \lambda b^k \end{bmatrix}$$

– Write

$$U_k = \begin{bmatrix} u_1^k \\ u_2^k \\ u_3^k \\ \vdots \\ u_{n-2}^k \\ u_{n-1}^k \end{bmatrix}, \; V_k = \begin{bmatrix} a^k \\ 0 \\ 0 \\ \vdots \\ 0 \\ b^k \end{bmatrix}$$

then

$$U_{k+1} = AU_k + \lambda V_k$$

where

$$A = \begin{bmatrix} 1-2\lambda & \lambda & \cdots & \cdots & \cdots & 0 \\ \lambda & 1-2\lambda & \lambda & \cdots & 0 & 0 \\ 0 & \lambda & 1-2\lambda & \lambda & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1-2\lambda & \lambda \\ 0 & 0 & 0 & \cdots & \lambda & 1-2\lambda \end{bmatrix} = I + \begin{bmatrix} -2\lambda & \lambda & \cdots & \cdots & \cdots & 0 \\ \lambda & -2\lambda & \lambda & \cdots & 0 & 0 \\ 0 & \lambda & -2\lambda & \lambda & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & -2\lambda & \lambda \\ 0 & 0 & 0 & \cdots & \lambda & -2\lambda \end{bmatrix}$$

$$= I + \lambda \begin{bmatrix} -2 & 1 & \cdots & \cdots & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix} = I + \lambda H$$

– Euler's scheme:

$$U_{k+1} = (I + \lambda H)\, U_k + \lambda V_k = U_k + \lambda H U_k + \lambda V_k$$

– This is explicit forward scheme. It is stable and consistant if $\lambda = hn^2 < 0.5$ (Courant–Friedrichs–Lewy (CFL) condition)

– Crank-Nicolson scheme:

$$U_{k+1} = U_k + \frac{1}{2}\lambda\,(HU_k + HU_{k+1}) + \lambda V_k$$

$$U_{k+1} = \left(I - \frac{\lambda}{2} H\right)^{-1} \left(U_k + \frac{1}{2}\lambda H U_k + \lambda V_k\right)$$

- Homework: 12.6, 12.8, 12.9