Chapter 11 Partial Differential Equations

- Review partial derivative/vector calculus: Let $f(x,y,z)$ be a function, $F(x,y,z) = (P(x,y,z), Q, R)$ be a vector field.
  - Gradient operator

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

  - Gradient of f

$$\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

  - Divergence of vector field F

$$\begin{aligned}\nabla \circ F &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \circ F(x,y,z) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \circ (P, Q, R) \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\end{aligned}$$

– The curl of F

$$\nabla \times F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times F(x,y,z) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

– The Laplacian $\Delta = \nabla \circ \nabla$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- Section11.2 Some PDEs

(a) Transport Equation for $u(t,x,y,z)$

$$\frac{\partial u}{\partial t} + \vec{c} \circ \nabla u = f$$

where $\vec{c}$ is a constant vector

(b) Conservation laws:

$$\frac{\partial u}{\partial t} + \nabla \circ \left(u\vec{C}\right) = 0$$

where $\vec{C}\left(x,y,z\right).\,If\ \vec{C}\left(x,y,z\right)=\vec{c}$, it becomes a transport.

(c) The heat equation

$$\frac{\partial u}{\partial t}=c^2\nabla^2 u=c^2\Delta u$$

(d) The wave equation

$$\frac{\partial^2 u}{\partial t^2}=c^2\Delta u$$

(e) Laplace's equation

$$\Delta u=0$$

(f) Poisson's equation

$$\Delta u=f$$

(g) Schrodinger's equation for $\psi=u\left(t,x,y,z\right)+iv\left(t,x,y,z\right):$

$$i\frac{\partial\psi}{\partial t}=-c^2\Delta\psi+a\left(x,y,z\right)\psi$$

(h) The Plate/beam equation (biharmonic)

$$\frac{\partial u}{\partial t} = -c^2 \Delta^2 u = -c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

(i) The Navier-Stokes equation

$$\frac{\partial u}{\partial t} + \frac{1}{\rho} \nabla p + (u \circ \nabla) u = \nu \Delta^2 u$$

$$\rho_t + \nabla \circ \rho u = 0$$

(j) The Euler equation: $\nu = 0$

$$\frac{\partial u}{\partial t} + \frac{1}{\rho} \nabla p + (u \circ \nabla) u = 0$$

$$\rho_t + \nabla \circ \rho u = 0$$

(k) The Maxwell equations for E (electric field) and B (magnetic field)

$$\nabla \circ E = \frac{\rho}{\varepsilon}$$
$$\nabla \times B = \mu \left(J + \varepsilon \frac{\partial E}{\partial t} \right)$$
$$\nabla \times E = -\frac{\partial B}{\partial t}$$
$$\nabla \circ B = 0$$

See: https://en.wikipedia.org/wiki/Maxwell%27s_equations

- Section 11.3 Separation of variables:
- Well-posedness: We consider the heat equation

$$\frac{\partial u}{\partial t} = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

If $u\left(t, x, y, z\right)$ is stationary, i.e., independent of time t,it reduce to the Laplace equation

$$0 = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

In one spatial dimension $u = u\,(t, x)$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

In two spatial dimension $u = (t, x, y)$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let Ω be a spatial domain in R^n, $n = 1, 2, 3$. We call Ω a rectangular domain if

$$\begin{aligned}\Omega &= (a, b) \times (c, d) \times (e, f) \\ &= \{(x, y, z) \ : \ a < x < b, c < y < d, e < z < f\} \quad \text{(3-D)}\end{aligned}$$

$$\begin{aligned}\Omega &= (a, b) \times (c, d) \\ &= \{(x, y, z) \ : \ a < x < b, c < y < d\} \quad \text{(2-D)}\end{aligned}$$

or $\Omega = (a, b)$ if in 1-D. The boundary of D is

$$\partial D = \{(x, y, z) \ : \ x = a, \ \text{or } x = b, \ \text{or } y = c, \ \text{or } y = d, \ or \ \ z = e, \ \text{or } z = f\}$$

If $\Omega = B_r(a,b,c) = ball$ centered at (a,b,c) with the radius r, then $\partial D =$ sphere.

- Cauchy problem: Let $\Omega = R^n$. A Cauchy problem is referred to the following initial value problem

$$\frac{\partial u}{\partial t} = \Delta u,\ u(0,x,y,z) = u_0(x,y,z)$$

- Boundary Value problem for Laplace's equation:

– Dirichlet Boundary Value problem:

$$\begin{aligned} \Delta u &= 0 \\ u(x,y,z) &= f(x,y,z) \text{ for } (x,y,z) \text{ on } \partial D \end{aligned}$$

– Neumann Boundary Value problem:

$$\begin{aligned} \Delta u &= 0 \\ \frac{\partial u(x,y,z)}{\partial \vec{n}} &= f(x,y,z) \text{ for } (x,y,z) \text{ on } \partial D \end{aligned}$$

where $\vec{n}(x,y,z) = the$ unit normal vector pointing to outside of Ω, and directional derivative is

$$\frac{\partial u(x,y,z)}{\partial \vec{n}} = \nabla u \circ \vec{n}$$

– Mixed (Robin) Boundary Value problem:

$$\Delta u = 0$$
$$\frac{\partial u(x,y,z)}{\partial \vec{n}} + \beta u(x,y,z) = f(x,y,z) \text{ for } (x,y,z) \text{ on } \partial D$$

- Initial-Boundary Value problem for the heat equation:

– Dirichlet Boundary Value problem:

$$\frac{\partial u}{\partial t} = \Delta u$$
$$u(t,x,y,z) = f(x,y,z) \text{ for } (x,y,z) \text{ on } \partial D$$
$$u(0,x,y,z) = g(x,y,z)$$

– Neumann Boundary Value problem:

$$\frac{\partial u}{\partial t} = \Delta u$$
$$\frac{\partial u(t,x,y,z)}{\partial \vec{n}} = f(x,y,z) \text{ for } (x,y,z) \text{ on } \partial D$$
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where $\vec{n}(x, y, z) = the$ unit normal vector pointing to outside of Ω, and directional derivative is

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– Mixed (Robin) Boundary Value problem:

$$\frac{\partial u}{\partial t} = \Delta u$$

$$\frac{\partial u(t, x, y, z)}{\partial \vec{n}} + \beta u(x, y, z) = f(x, y, z) \text{ for } (x, y, z) \text{ on } \partial D$$

$$u(0, x, y, z) = g(x, y, z)$$

- Separation of variable for The heat equation in a rectangular domain in 1D: $u(t, x) = T(t) X(x)$
- Transport Equation: Let $u = f(x - c_1 t, y - c_2 t, z - c_3 t)$ .Then

$$u_t = -c_1 f_x - c_2 f_y - c_3 f_z = -\vec{c} \circ \nabla f, \quad \nabla u = \nabla f$$

$$u_t + \vec{c} \circ \nabla u = -\vec{c} \circ \nabla f + \vec{c} \circ \nabla f = 0$$

So for any f, $u = f(x - c_1 t, y - c_2 t, z - c_3 t)$ is a solution. Any solution is constant along

$$(x, y, z) = \vec{c}t$$

- D'Alembert's formula for wave equation in 1-D

$$\frac{\partial^2 u(t,x)}{\partial t^2} = c^2 \frac{\partial^2 u(t,x)}{\partial x^2}$$

Changing variables

$$\xi = x - ct$$
$$\delta = x + ct$$

$$w(\xi, \delta) = u(t,x)$$

then

$$\begin{aligned} u_t &= w_\xi(-c) + w_\delta c,\ u_x = w_\xi + w_\delta \\ u_{tt} &= (w_\xi(-c) + w_\delta c)_t \\ &= -c(w_{\xi\xi}(-c) + w_{\xi\delta} c) + c(w_{\delta\xi}(-c) + w_{\delta\delta} c) \\ &= c^2 w_{\xi\xi} - 2c^2 w_{\xi\delta} + c^2 w_{\delta\delta} \end{aligned}$$

$$\begin{aligned} u_x &= w_\xi + w_\delta \\ u_{xx} &= w_{\xi\xi} + 2w_{\xi\delta} + w_{\delta\delta} \end{aligned}$$

Substituting into the wave equation, we find

$$\left(c^2 w_{\xi\xi} - 2c^2 w_{\xi\delta} + c^2 w_{\delta\delta}\right) = c^2 \left(w_{\xi\xi} + 2w_{\xi\delta} + w_{\delta\delta}\right)$$

$$-2c^2 w_{\xi\delta} = 2c^2 w_{\xi\delta} \implies w_{\xi\delta} = 0$$

$$w = f\left(\xi\right) + g\left(\delta\right) = f\left(x - ct\right) + g\left(x + ct\right)$$

- Another approach:

$$D_t = \frac{\partial}{\partial t}, \quad D_x = \frac{\partial}{\partial x}$$

So

and the wave equation is

$$\left(D_t + cD_x\right)\left(D_t - D_x\right)u = \left(D_t^2 - c^2 D_t^2\right)u = u_{tt} - c^2 u_{xx} = 0$$

Note both

$$\begin{aligned} \left(D_t - D_x\right)u &= 0 \\ \left(D_t + cD_x\right)u &= 0 \end{aligned}$$

are transport equations with velocity $c$and $-c$.So general solutions are, respectively,

$$f(x+ct)\ ,\ g(x-ct)$$

Consider, for instance $c=1, \Omega=(0,1)$.The boundary condition at $x=0$ and $x=1$ must satisfy a "reflective condition" to avoid conflict. In other words, there are compatibility condition needed to coordinate initial and boundary conditions.

- Homework: 11.6, 11.8, 11.9