Chapter 9 Global Nonlinear Techniques

Consider nonlinear dynamical system

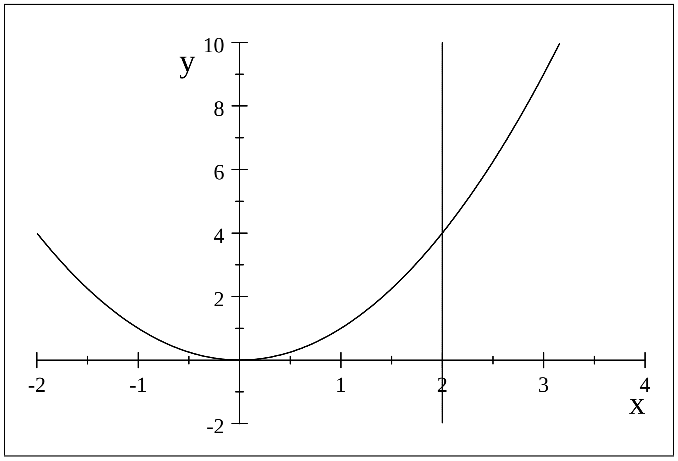
$$X' = F(X) = \begin{pmatrix} f_1(X) \\ f_2(X) \\ \vdots \\ f_n(X) \end{pmatrix}$$

- Nullcline
  - $x_j - nullcline = \{X : f_j(X) = 0\}$
  - equilibrium solutions = intersection of all $x_j - nullclines$
  - $x_j - nullcline$ is usually a surface of co-dimension one
  - total N $x_j - nullcline$, $j = 1, ..., n$, divide the space into at least 2^n parts
  - type of equilibrium can be easily figured out by looking at one direction in each part
  - In R^2, there are two nullcline curves. They divide the plane into four parts. Each will be one of the followings
    * NE (in which all directions in the directional field point to northeast)
    * NW (in which all directions in the directional field point to northwest)
    * SE (in which all directions in the directional field point to southeast)
    * SW (in which all directions in the directional field point to southwest)

Example 1 Consider (p. 189)

$$x' = y - x^2$$
$$y' = x - 2$$

- $x - nullcline$: parabola $y = x^2$
- $y - nullcline$: vertical line $x = 2$



- there is only one equilibrium $X_0 = (2, 4)$
- the line $x = 2$ and the parabola $y = x^2$ divide the plane into four parts.
  - In part A containing $X = (0, 4)$, $F(X) = (4, -2)$ (SE)
  - In part B containing $X = (3, 10)$, $F(X) = (1, 1)$ (NE)
  - In part C containing $X = (3, 0)$, $F(X) = (-9, 1)$ (NW)
  - In part D containing $X = (1, 0)$, $F(X) = (-1, -1)$ (SW)
- X_0 is a saddle:
  - Solutions with initial values in part B and D will stay in the same regions and move away from the origin
  - Solutions with initial values in part A and C will either enter B or D, or remain in the same region. In the latter case, the solutions will converge to X_0.(stable curve). All other solutions once enter B or D, they will stay there and move away from the origin.
  - Its linearization at $X_0 = (2, 4)$ is

$$x' = -4x + y$$
$$y' = x$$

  - For $A = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix}$, the eigenvalues are $\lambda = -2 \pm \sqrt{5}$. For $\lambda = -2 - \sqrt{5}$, one eigenvector is $X = \left(1, 2 - \sqrt{5}\right) = (1, -0.236)$. So the stable curve passes through X_0 with the tangent parallel to the direction $(1, -0.236)$

- Recall **Stability:**
  - An equilibrium X_0 is called **stable** if any solution with initial value X_1 near X_0 will remain nearby.
  - For linear system, sinks, spiral sinks, and centers are stable.
  - An equilibrium X_0 is called **asymptotically stable** if it is stable and it converges to X_0 as $t \to \infty$.
  - Stability Theorem: Any equilibrium that is a sink or spiral sink for its linearization is asymptotically stable.
  - In other words, an equilibrium X_0 is asymptotically stable if all eigenvalues of $DF(X_0)$ have negative real parts.
- We shall extend this to the case when some eigenvalues have zero real parts.
- Liapunov Stability Analysis
  - Liapunov function for the dynamical system around X_0 : A function defined in a neighborhood O of X_0 satisfying

    a $L(X_0) = 0,\ L(X) > 0$ for $X \neq X_0,\ X \in O$

    b $\nabla L(X) \cdot F(X) \leq 0$ for all $X \in O$
  - **Liapunov Stability Theorem**: Let X_0 be an equilibrium for $X = F(X)$. The equilibrium is stable if there exists a smooth Liapunov function. If furthermore, the Liapunov function satisfies

    c $\nabla L(X) \cdot F(X) < 0$ for $X \neq X_0$

    then X_0 is asymptotically stable (i.e., $X(t) \to X_0$ as $t \to \infty$)
- Note that if all eigenvalues of $DF(X_0)$ are negative, then we define

$$L(X) = -\frac{1}{2}(X - X_0)^T DF(X_0)(X - X_0) \geq 0$$

one can verify this is a Liapunov function. This is because $DF(X_0)$ is negatively definite, and, by Taylor expansion

$$\begin{aligned} F(X) &= F(X_0) + DF(X_0)(X - X_0) + O\left(\|X - X_0\|^2\right) \\ &= DF(X_0)(X - X_0) + O\left(\|X - X_0\|^2\right) \end{aligned}$$

and

$$\nabla L(X) = -DF(X_0)(X - X_0).$$

So

$$\nabla L(X)\cdot F(X) = -(DF(X_0)(X - X_0))\cdot(DF(X_0)(X - X_0)) + O\left(\|X - X_0\|^3\right) < 0$$

as $X \to X_0$. Liapunov Theorem includes Stability Theorem.

Example 2. Consider, for parameter ε,

$$\begin{aligned} x' &= (\varepsilon x + 2y)(z + 1) \\ y' &= (-x + \varepsilon y)(z + 1) \\ z' &= -z^3 \end{aligned}$$

The only equilibrium is $X_0 = 0$, and its linearization is

$$\begin{aligned} x' &= \varepsilon x + 2y \\ y' &= -x + \varepsilon y \\ z' &= 0 \end{aligned}$$

with eigenvalues $0, \varepsilon \pm i\sqrt{2}$. So this is not hyperbolic. Therefore, the linearization does not indicate anything about the nonlinear system. For $\varepsilon < 0$, we look for a Liapunov function in the form:

$$L(x, y, z) = ax^2 + by^2 + cz^2$$

We see that

$$\begin{aligned} \nabla L \cdot F &= 2(ax, by, cz) \cdot F \\ &= 2ax(\varepsilon x + 2y)(z + 1) + 2by(-x + \varepsilon y)(z + 1) + 2cz\left(-z^3\right) \\ &= 2\varepsilon\left(ax^2 + by^2\right)(z + 1) + (2a - b)\, yx\,(z + 1) - 2cz^4 \end{aligned}$$

So if we choose $a = 1, b = 2, c = 1$. Then for all X

$$\nabla L \cdot F(X) = 2\varepsilon\left(ax^2 + by^2\right)(z+1) - 2cz^4 \leq 0$$
$$\nabla L \cdot F(X) < 0 \text{ if } X \neq 0$$

Hence, the equilibrium $X_0 = 0$ is asymptotically stable. However, for $\varepsilon = 0$, we can only conclude that the equilibrium is stable.

- Justification of Liapunov Theorem:
  - For any solution $X(t)$ of the system $X' = F(X(t))$, we have (by (b))

    $$\frac{d}{dt}L(X(t)) = \nabla L(X(t)) \cdot X'(t) = \nabla L(X(t)) \cdot F(X(t)) \leq 0$$

  - So $L(X(t))$ decreases to $L(X(0))$
  - For any $\alpha > 0$, the set $G = \{X : L(X) < \alpha\}$ is a neighborhood of X_0.
  - For any $X_0 \in G$, *along* the solution $X(t)$ initiated from X_0,since $L(X(t))$ decreases to $L(X_0)$

    $$L(X(t)) \leq L(X_0) < \alpha$$

  - So the entire solution $X(t)$ remains in G — stable

Example 3. (Nonlinear pendulum) Consider a pendulum consisting of a light rod of length l to which is attached a ball of mass m.The other end of the rod is attached to a point on the ceiling. The position of the mass is described by the angle $\theta(t)$ from the straight-down position and measured in the counterclockwise direction. So the position of the mass is $l(\sin\theta(t), -\cos\theta(t))$, and velocity and acceleration are, respectively

$$v = l(\cos\theta, \sin\theta)\theta'$$
$$a = v' = l(\cos\theta, \sin\theta)\theta'' + l(-\sin\theta, \cos\theta)\theta'$$
$$F = (0, -mg) - bl(\cos\theta, \sin\theta)\theta'$$

We assume the only forces are gravitational force and Stoke's friction to be proportional to its velocity:

$$\begin{aligned} F &= (0, -mg) - blv \\ &= (0, -mg) - bl(\cos\theta, \sin\theta)\,\theta'. \end{aligned}$$

Now Newton's law $F = ma$ along the tangential direction $(\cos\theta, \sin\theta)$ leads to

$$ma \cdot (\cos\theta, \sin\theta) = F \cdot (\cos\theta, \sin\theta)$$

or nonlinear Pendulum model:

$$ml\theta'' + bl\theta' + mg\sin\theta = 0$$

Set $l = g = m = 1$In the system form, it is

$$\begin{aligned} \theta' &= v \\ v' &= -bv - \sin\theta. \end{aligned}$$

The total energy functional can be used as its Liapunov function:

$$E(\theta, v) = \frac{1}{2}v^2 + 1 - \cos\theta$$

One can verify that, for $b \geq 0$,

$$\nabla E(\theta, v) \cdot F = (\sin\theta, v) \cdot (v, -bv - \sin\theta) = -bv^2 \leq 0$$

Thus the equilibrium $\theta = 0$, $v = 0$ is stable.

Example 4. Show $(0, 0)$ is asymptotically stable for

$$\begin{aligned} x' &= -\frac{1}{2}x + x^2 + 2y^2 \\ y' &= -x - y + 2x^2 \end{aligned}$$

Sol: We try $L = ax^2 + by^2$ for some $a, b > 0$.

$$\begin{aligned} \nabla L &= (2ax, 2by) \\ \nabla L \cdot F &= 2ax\left(-\frac{1}{2}x + x^2 + 2y^2\right) + 2by\left(-x - y + 2x^2\right) \\ &= -ax^2 + 2ax^3 + 4axy^2 - 2bxy - 2by^2 + 4bx^2y \\ &= -\left(ax^2 + 2by^2 + 2bxy\right) + 2ax^3 + 4axy^2 + 4bx^2y. \end{aligned}$$

Choose $b = 1$, $a = 2$ we have

$$\nabla L \cdot F = -\left(x^2 + y^2 + 2xy\right) - x^2 - y^2 + 2ax^3 + 4axy^2 + 4bx^2 y$$
$$= -(x+y)^2 - x^2(1 - 4x - 4y) - y^2(1 - 8x)$$

When $|x| < 1/8, |y| < 1/8$,

$$\nabla L \cdot F < 0.$$

Therefore, it is asymptotically stable.

- Some special nonlinear systems

  1. Gradient Flows: $V(X)$ is a smooth function $R^n \to R^1$

  $$X' = -\nabla V(X)$$

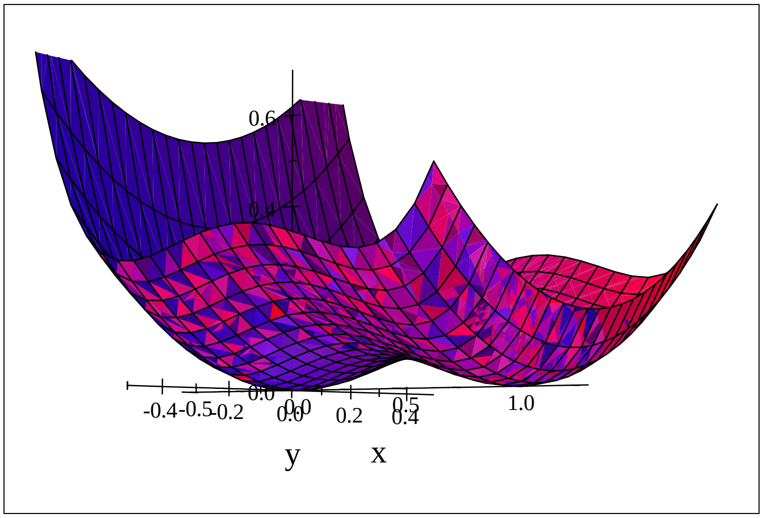
     - For any solution $X(t)$,

     $$\frac{d}{dt} V(X(t)) = \nabla V(X(t)) \cdot X'(t) = -|\nabla V(X(t))|^2 \leq 0$$

     - So the potential function V decreases along any solution curve.
     - For any level surface $\{X : V(X) = c\}$ of V, its tangent plane has the normal direction ∇V
     - Therefore, any solution curve is moving towards lower-value level surfaces and is perpendicular to level surface
     - Any solution $X(t)$ will approach to a point that reaches a minimal value of V
     - Critical points X_0 (i.e., $\nabla V(X_0) = 0$) of V are equilibrium solutions.
     - All equilibria are stable.
     - Any isolated local minimum points X_0 are asymptotically stable.
     - Linearization matrices are symmetric, and have only real eigenvalues.

  Example 5 Consider gradient system

  $$V = x^2(x-1)^2 + y^2$$



Since

$$\nabla V = \left(2x\left(2x^2 - 3x + 1\right),\ 2y\right)$$

there are three critical points: $X_0 = (0,0)\,,\ (1,0)\,,\ (0.5,0)$ .The first two are local minimums of V. But $(0.5,0)$ is not.

2. Hamiltonian Systems: $H(X)$ is smooth function (called Hamiltonian function)

$$x' = \frac{\partial H}{\partial y}$$

$$y' = -\frac{\partial H}{\partial x}$$

- $H(X)$ is constant along any solution $X(t)$

$$\frac{d}{dt}H(X(t)) = \nabla H(X(t)) \cdot X'(t) = (\partial_x H, \partial_y H) \cdot (\partial_y H, -\partial_x H) = 0$$

$$H(X(t)) = H(X(0))$$

- thus, Hamiltonian system is conservative system: it will not alter value of H
- Therefore, a solution curve is a part of a level curve of H
- linearizations have the structure

$$A = \begin{pmatrix} \dfrac{\partial^2 H}{\partial x \partial y} & \dfrac{\partial^2 H}{\partial y^2} \\ -\dfrac{\partial^2 H}{\partial x^2} & -\dfrac{\partial^2 H}{\partial x \partial y} \end{pmatrix}$$

- Its characteristic polynomial is

$$\lambda^2 - \left(\frac{\partial^2 H}{\partial x \partial y}\right)^2 + \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2} = 0$$

- Eigenvalues of any linearized Hamiltonian system at critical point $\nabla H(X_0) = 0$:
  * If $\det(D^2 H) = \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2} - \left(\frac{\partial^2 H}{\partial x \partial y}\right)^2 > 0$ (local minimum of H), then eigenvalue are $\pm\sqrt{\det(D^2 H)}i$, and thus X_0 is a center.
  * If $\det(D^2 H) < 0$ (local saddle of H), then eigenvalue are $\pm\sqrt{-\det(D^2 H)}$,and thus X_0 is a saddle.

Example 6. (ideal pendulum) The frictionless pendulum

$$\begin{aligned} \theta' &= v \\ v' &= -\sin\theta \end{aligned}$$

is a Hamiltonian with

$$H(\theta, v) = \frac{1}{2}v^2 + 1 - \cos\theta$$

Example 7. Consider

$$\begin{aligned} x' &= y \\ y' &= -x^3 + x \end{aligned}$$

This is a Hamiltonian with

$$H = \frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}$$

There are three equilibrium solutions $X_0 = (0,0), (\pm 1, 0)$. Note that the linearized system is $X' = AX$

$$A = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix}$$

while the Hessian matrix is

$$D^2 H = \begin{pmatrix} 3x^2 - 1 & 0 \\ 0 & 1 \end{pmatrix}$$

At $(0,0)$, $D^2 H$ is neither positive nor negative, so $(0,0)$ is a saddle for H, or a saddle for the linearized system $X' = A(0,0)X$. At the other two equilibria $X_0 = (\pm 1, 0)$, $D^2 H$ is positively definite. So they are local minimum points of H, and are center for the linearized systems.

- Homework: 1ab, 2, 6, 7aef (no phase portrait, no level surface)
- #8 (no phase portrait. Do the followings)
    1. Determine which is gradient, which is Hamiltonian.
    2. (optional) If it is a gradient or Hamiltonian, find V or H.

    – Hints for #8: In 2d, a system

    $$x' = f$$
    $$y' = g$$

    a is a gradient iff

    $$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

    then, V can be found by solving

    $$\frac{\partial V}{\partial x} = f$$
    $$\frac{\partial V}{\partial y} = g$$

    b is a Hamiltonian iff

    $$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$$