Chapter 8 Equilibria in Nonlinear Systems

- Recall linearization for Nonlinear dynamical systems in R^n:

$$X' = F(X).$$

  - if X_0 is an equilibrium, i.e., $F(X_0) = 0$, then its linearization is

$$U' = AU, \quad A = DF(X_0)$$

  - Let $X(t)$ is the nonlinear solution satisfying $X(0) = X_0 + U_0$
    * if $U(t)$ is the solution to its linearization satisfying $U(0) = U_0$
    * then $X_0 + U(t)$ is an approximation to $X(t)$ of order more than one:

$$\frac{|X(t) - (X_0 + U(t))|}{|U_0|} \to 0 \quad \text{as } |U_0| \to 0$$

    * But this doesn't mean nonlinear system and its linearization are structurally the same near X_0, i.e., they may be or may not be conjugate
  - Example 1.

$$\begin{aligned} x' &= x + y^2 \\ y' &= -y. \end{aligned}$$

The flow is

$$\phi(t; x_0, y_0) = \begin{pmatrix} \left(x_0 + \frac{1}{3}y_0^2\right)e^t - \frac{1}{3}y_0^2 e^{-2t} \\ y_0 e^{-t} \end{pmatrix}$$

If initially,

$$\left(x_0 + \frac{1}{3}y_0^2\right) = 0$$

then the solution approaches to the equilibrium $(0, 0)$.

The nonlinear system has an equilibrium solution $(x, y) = (0, 0)$. The linearization near $(0, 0)$ is

$$x' = x$$
$$y' = -y.$$

The flow for the linearization is

$$\psi(t; x_0, y_0) = \begin{pmatrix} x_0 e^t \\ y_0 e^{-t} \end{pmatrix}$$

If initially $x_0 = 0$, all solutions approach to the equilibrium $(0, 0)$. In this case, both nonlinear system and its linearization has a "stable" manifold: the former is a parabola and the latter is $y-axis$. And, in this case near $(0, 0)$, we can simply "drop off" the nonlinear term y^2 in both system and in solution (drop y_0^2).

But this is not always the case.

- Example 2. Consider

$$x' = x^2$$
$$y' = -y.$$

  * Solutions are
$$x = \frac{x_0}{1 - x_0 t}, \quad y = y_0 e^{-t}$$
  * For $x_0 < 0$, solution $(x, y) \to (0, 0)$ from the left, as $t \to \infty$
  * For $x_0 > 0$, but x_0 very small, the solution exists up to $0 \leq t < 1/x_0$, and $x \to \infty$ from the left as $t \to 1/x_0 -$.
  * However, its linearization
$$x' = 0$$
$$y' = -y.$$
    has solution
$$x = x_0, \quad y = y_0 e^{-t}$$
  * So x components for these two systems are structurally different.

- Comparing these two examples: the equilibrium X_0 in Example is hyperbolic, while in the second example, it is not hyperbolic

- **Hyperbolic equilibrium** X_0

  - As we discussed before, two linear systems are conjugate if they have the same numbers of eigenvalues with negative real parts and positive real parts.
  - So in this case the linearization is conjugate to a linear system with distinct real eigenvalues.
  - Assuming $X_0 = 0$ (otherwise by a translation), and $DF(X_0)$ has distinct real eigenvalues, k negative eigenvalues and $(n-k)$ positive.
  - Let A be its linearization:

$$X' = F(X) = AX + G(X)$$

$$\begin{aligned} \text{eigenvalue}(A) &= \{-\lambda_1, ..., -\lambda_k, \mu_1, ..., \mu_{n-k}\}, \\ &0 < \ \lambda_1 < \lambda_2 < ... < \lambda_k, \quad \mu_j > 0 \end{aligned}$$

$$|G(X)| \, |X|^{-1} \to 0 \ \text{ as } \ |X| \to 0$$

  - If X_0 is a sink, i.e., $k = n$
  - In general, when $k < n$, Write

$$S = span\{\text{all eigenvectors with negative eigenvalues}\}.$$

  - Then, for any initial value $X_0 \in S$ and $|X_0|$ is small, the solution $X(t)$ is moving towards the origin. This is because the tangent direction of the solution $X'(t)$ is opposite to X :

$$\begin{aligned} X' \cdot X &= (AX + G(X)) \cdot X = X^T AX + G(X) \cdot X \\ &= X^T AX + G(X) \cdot X \\ &\leq \left(-\lambda_1 + |G(X)| \, |X|^{-1}\right) |X|^2 < 0 \quad \text{if } |X| \text{ is small} \end{aligned}$$

    So

$$\frac{d}{dt} \|X(t)\|^2 = \frac{d}{dt}(X \cdot X) = 2X'(t) \cdot X(t) < 0$$

    Hence $\|X(t)\|^2$ is decreasing with a pure negative rate.

- So when X_0 is sink, all solutions are asymptotically stable, i.e., $X(t) \to 0$.

- **Stable manifold:** In general, if there are k eigenvalues with negative real part, then there is a $k-$ dimensional stable manifold (locally) with tangent space S, in the sense that any solution initiating from it (and close to the origin) will stay there and the solution converges to the origin.

- In 2D case, if

$$\begin{aligned} x' &= \mu x + f_1(x, y) \\ y' &= -\lambda y + f_2(x, y) \end{aligned}$$

then there is a stable curve $x = h(y)$ that passes through the origin and is tangent to the $y - axis : h(0) = 0,\ h'(0) = 0$.

- **Linearization Theorem**: Suppose that $A = DF(X_0)$ is hyperbolic, i.e., all eigenvalues have non-zero real parts. Then nonlinear flow is conjugate to the linear flow of its linearization system near X_0 locally.

Example 3. Let us reconsider Example 1

$$\begin{aligned} x' &= x + y^2 \\ y' &= -y \end{aligned}$$

The solution for the second equation is

$$y = y_0 e^{-t}.$$

Substituting into the first equation

$$x' = x + y_0^2 e^{-2t},$$

we see solution for the nonlinear system:

$$\begin{aligned} x &= \left(x_0 + \frac{y_0^2}{3}\right) e^t - \frac{y_0^2}{3} e^{-2t} \\ y &= y_0 e^{-t} \end{aligned}$$

If initially $x_0 + \frac{y_0^2}{3} = 0$, then the solution is

$$x = x_0 e^{-2t}$$
$$y = y_0 e^{-t}$$

which will go to the origin as $t \to \infty$. So the stable curve is $x + \frac{y^2}{3} = 0$.

On the other hand, its linearization is

$$x' = x$$
$$y' = -y$$

The solution for the linearization is

$$x = x_0 e^{t}$$
$$y = y_0 e^{-t}$$

- We now demonstrate these two systems are globally conjugate.
  - Introduce new variables $(u, v) = h(x, y) = (h_1(x, y), h_2(x, y))$

$$u = x + \frac{y^2}{3},$$
$$v = y.$$

  - Under the new variables, the nonlinear system becomes:

$$u' = x' + \frac{2}{3}yy' = \left(x + \frac{y^2}{3}\right) + \frac{2}{3}y(y) = u$$
$$v' = y' = -y = -v$$

  - So if (x, y) is a nonlinear solution, then (u, v) solves the linearized system:

$$u' = u$$
$$v' = -v$$

- Now, for nonlinear solution (x, y) :

$$x = \left(x_0 + \frac{y_0^2}{3}\right) e^t - \frac{y_0^2}{3} e^{-2t}$$
$$y = y_0 e^{-t}$$

- we see that under new variables

$$\begin{aligned} u = h_1(x, y) &= x + \frac{y^2}{3} \\ &= \left(x_0 + \frac{y_0^2}{3}\right) e^t - \frac{y_0^2}{3} e^{-2t} + \frac{1}{3}\left(y_0 e^{-t}\right)^2 \\ &= \left(x_0 + \frac{y_0^2}{3}\right) e^t \\ &= h_1(x_0, y_0) e^t \end{aligned}$$
$$v = h_2(x, y) = y = y_0 e^{-t} = h_2(x_0, y_0) e^{-t}$$

- So if $\phi(t, X_0)$ is the nonlinear flow, and $\psi(t, U_0)$ is the linear flow, then

$$h(\phi(t, X_0)) = \psi(t, h(X_0))$$

- hence, $h(x, y)$ is a homomorphism that maps solutions of nonlinear system to its linearization, i.e., they are conjugate.
- The stable manifold for the linearization is $u = 0$
- The stable manifold for the nonlinear system is $h_1(x, y) = x + y^2/3 = 0$.

In general, it is not always possible to find such a global conjugacy.

Example 4: Nonlinear system (page 163)

$$x' = \frac{1}{2}x - y - \frac{1}{2}x\left(x^2 + y^2\right)$$
$$y' = x + \frac{1}{2}y - \frac{1}{2}y\left(y^2 + x^2\right).$$

For any functions $(x(t), y(t))$ satisfying $x^2 + y^2 = 1$, we see

$$x' = \frac{1}{2}x - y - \frac{1}{2}x = -y$$
$$y' = x + \frac{1}{2}y - \frac{1}{2}y = x$$

and thus a periodic solution

$$x = \cos t$$
$$y = \sin t.$$

Introducing polar coordinates (r, θ), under the polar coordinates, the system becomes

$$r' = \frac{1}{2} r \left(1 - r^2\right)$$
$$\theta' = 1.$$

So if initially $r_0 = \sqrt{x_0^2 + y_0^2} > 1$, then $r' < 0$, so $r(t)$ is decreasing to $r = 1$. Otherwise if initially $r < 1$, $r' > 0$ and r increases to $r = 1$. In other words, the unit circle is an attractor: any solution tends to the unit circle. On the other hand, its linearization

$$x' = \frac{1}{2} x - y$$
$$y' = x + \frac{1}{2} y$$

has eigenvalue $\frac{1}{2} \pm i$, so all solutions spiral away from the origin. In this case, it is impossible to find a global conjugacy. It can only be defined near the origin.

Example 5. Consider

$$x' = -x$$
$$y' = -y$$
$$z' = z + x^2 + y^2$$

has eigenvalues $-1, -1, 1$. The change of variables

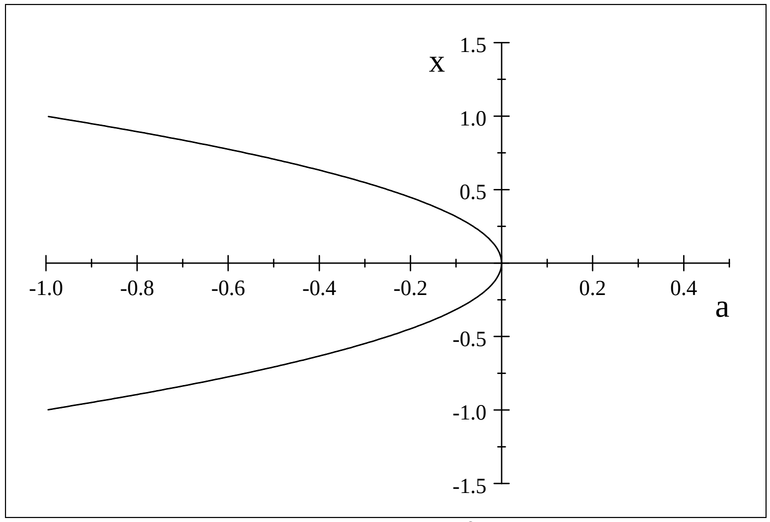
$$u = x, \ v = y, \ w = z + \frac{1}{3} \left(x^2 + y^2\right)$$

converts the nonlinear system to its linearized system

$$u' = -u$$
$$v' = -v$$
$$w' = w$$

So the stable manifold is $w = z + \frac{1}{3}(x^2 + y^2) = 0$, which is paraboloid opening downward.

- **Stability**
  - An equilibrium X_0 is called **stable** if for every neighborhood G of X_0, there is a neighborhood $G_1 \subset G$ of X_0 such that any solution initiated from inside G_1 will remain in G for all t.
  - For linear system, sinks, spiral sinks, and centers are stable.
  - An equilibrium X_0 is called **asymptotically stable** if it is stable and it converges to X_0 as $t \to \infty$.
  - For linear system, sinks and spiral sinks are asymptotically stable. The centers are stable, but not asymptotically stable.
- Stability Theorem: Any equilibrium that is a sink or spiral sink for its linearization is asymptotically stable.
- What we discussed above shows that Non-hyperbolic equilibrium solutions are only "partially stable", .i.e., there is a stable manifold.
- **Bifurcations:** For non-hyperbolic equilibrium solutions, there will be structural change. This is when bifurcation happens.
- $x' = f_a(x)$ (one dimension)
  - Saddle-Node bifurcation in 1d: a bifurcation point $a = a_0$ is called saddle-node type if
    * $a = a_0$, it has only one equilibrium point (often saddle)
    * On one side of a_0 (for instance, $a < a_0$), it has two or more equilibria
    * On the other side of a_0 (for instance, $a > a_0$), no equilibrium
  - Example: $x' = x^2 + a$

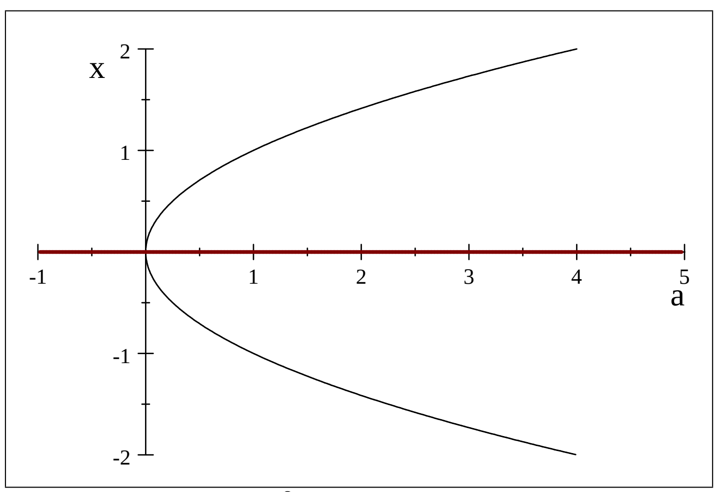


Example: $x' = x^2 + a$

- Theorem: $x' = f_a(x)$ undergoes a saddle-node bifurcation if

$$f_{a_0}(x_0) = f'_{a_0}(x_0) = 0$$
$$f''_{a_0}(x_0) \neq 0, \quad \frac{\partial}{\partial a} f_a(x_0) \neq 0.$$

- Pitchfork Bifurcation: $x' = x^3 - ax$



Bifurcatio diagram $x^3 - ax = 0$ looks like a pitchfork

  * for $a \leq 0$, $x = 0$ is the only equilibrium (source, since $f'_a(0) = -a >$)

  - * for $a > 0$, there are three equilibria $x = 0$ (sink), $x = \pm\sqrt{a}$ (source)

- Saddle-Node bifurcation in 2d: Consider

$$x' = x^2 - a$$
$$y' = -y$$

For $a = 0$, it has one equilibrium $(0, 0)$. For $a < 0$, no equilibrium. For $a > 0$, there are two $(\pm\sqrt{a}, 0)$. The Jacobian is

$$\begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix}$$

So at $(\pm\sqrt{a}, 0)$, the linearization is

$$\begin{pmatrix} \pm 2\sqrt{a} & 0 \\ 0 & -1 \end{pmatrix}$$

So for $(\sqrt{a}, 0)$, one saddle and on sink. For $(-\sqrt{a}, 0)$, both are sink.

- Example: System in polar coordinate system

$$r' = r - r^3$$
$$\theta' = \sin^2\theta + a$$

Find bifurcation point

Sol: $r - r^3 = 0$ has roots $r = 0$, $r = \pm 1$. Now, if $a > 0$, there is no equilibrium solution. If $a = 0$,equilibrium solutions are $\sin\theta = 0$. If $-1 < a < 0$, there are equilibrium solution $\sin^2\theta = -a$. (see figure 8.9 in page 181 for local behavior)

- **Hopf Bifurcation**: as a passes through a_0,there is no change of the number of equilibria. But there is a structural change. For instance, periodic solutions occur at $a = a_0$,while there is no periodic solution for $a \neq a_0$

  - * Example: Consider

$$x' = ax - y - x\left(x^2 + y^2\right)$$
$$y' = x + ay - y\left(x^2 + y^2\right)$$

- Its linearization at $(0,0)$ is

$$x' = ax - y$$
$$y' = x + ay$$

- The eigenvalues are $a \pm i$.
- $a > 0$ (spiral source)
- $a < 0$ (spiral sink)
- $a = 0$ is a bifurcation (center)
- As we discussed in a previous example, when $a = 0$, the unit circle is a periodic solution that is an attractor.
- All solution initiated outside the unit circle will converges to the unit circle

• Homework: #1ac(i, iii, iv), 2 (Only do the following: find the stable manifold at $(0,0,0)$), 5, 9