Chapter 7 Nonlinear Systems

Nonlinear systems in R^n:

$$X' = F(t, X)$$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \; F(t, X) = \begin{pmatrix} F_1(t, x_1, ..., x_n) \\ \vdots \\ F_n(t, x_1, ..., x_n) \end{pmatrix}$$

When $F(t, X) = F(X)$ is independent of t, it is an example of dynamical system.

- Dynamical System:
  - Definition: Consider any smooth two-variable function $\phi(t, X)$: $(t, X) \in R^1 \times R^n \longmapsto R^n$. Let $\phi_t(X) = \phi(t, X)$ be a map $R^n \longmapsto R^n$. We call the family of map $\{\phi_t\}$ a smooth dynamical system if it satisfies the following two conditions
    1. ϕ_0 is an identity map: $\phi_0(X) = \phi(0, X) = X$ for any X
    2. $\phi_t \circ \phi_s = \phi_{t+s}$: $\quad \phi(t, \phi(s, X)) = \phi(t + s, X)$
  - Flows of linear systems $X' = AX$ are dynamical systems
    * Solution is
    $$\phi_t(X_0) = \phi(t, X_0) = e^{tA} X_0$$
    * So
    $$\phi_0(X_0) = e^{0A} X_0 = X_0$$
    $$\phi_t \circ \phi_s(X_0) = \phi_t\left(e^{sA} X_0\right) = e^{tA} e^{sA} X_0 = e^{(t+s)A} X_0 = \phi_{t+s}(X_0)$$

- Flows of nonlinear systems $X' = F(X)$ are also dynamical systems (if F is smooth function)
  - Integral equation formulation:
  $$X(t) = X_0 + \int_0^t F(X(s))\, ds$$

- Picard Iteration: To solve this system, we create an iterative sequence

$$X_0 = X_0 \quad \text{(initial value)}$$
$$X_{n+1} = X_0 + \int_0^t F(X_n(s))\, ds$$

- Existence (for general system $F(t,X)$):
  - the Picard sequence $\{X_n(t)\}$ has a limit as $n \to \infty$,for a small time period $-\varepsilon < t < \varepsilon$. (local existence theorem)
  - the limit $X(t)$ is a solution for t in $(-\varepsilon, \varepsilon)$, because it satisfies the integral formulation
  - For linear (nonautonomous system)

    $$X' = A(t)X$$

    the Picard sequence converges for all t as long as $A(t)$ is defined and continuous. (Global existence)
- Uniqueness (for general system $F(t,X)$): The solution is unique, i.e., there is only one solution for the same IVP.
- Continuous dependence on initial data (for general system $F(t,X)$):
  - For Y_0 close to X_0, solution $Y(t)$ with the initial data Y_0 also exits in $(-\varepsilon, \varepsilon)$
  - From integral formulations

    $$Y(t) = Y_0 + \int_0^t F(Y(s))\, ds$$

    we see

    $$X(t)-Y(t) = X_0+\int_0^t F(X(s))\, ds-\left(Y_0 + \int_0^t F(Y(s))\, ds\right)$$

    So if $|\nabla F| \leq K$,

    $$\begin{aligned} |X(t) - Y(t)| &\leq |X_0 - Y_0| + \int_0^t |F(X(s)) - F(Y(s))\, ds| \\ &\leq |X_0 - Y_0| + \int_0^t |F(X(s)) - F(Y(s))\, ds| \\ &\leq |X_0 - Y_0| + K \int_0^t |X(s) - Y(s)|\, ds \end{aligned}$$

- * By Gronwall's inequality, we arrive at

$$|X(t) - Y(t)| \leq |X_0 - Y_0| e^{Kt}$$

  * So if $Y_0 \to X_0$,then $Y(t) \to X(t)$

- Continuous dependence on parameters (for general system $F(t, X)$):

$$X' = F_a(X)$$

  * Let $X_a(t)$ be the solution, and $X_b(t)$ the solution for $X' = F_b(X)$. Then

$$X_a(t) - X_b(t) = \int_0^t (F_a(X_a(s)) - F_b(X_b(s)))\, ds$$

  * So if $F_b(X) \to F_a(X)$ as $b \to a$, $X_b(t) \to X_a(t)$

- So the flow $\phi(t, X)$ exists and unique, and continuously depends on initial data X.
- The flow is a dynamical system:
  * By definition, $\phi(0, X) = X$
  * For any fixed s, set $Y(t) = \phi(t + s, X)$.Then $Y(t)$ solves

$$\begin{aligned} Y'(t) &= \frac{\partial}{\partial t}\phi(t+s, X) = F(\phi(t+s, X)) = F(Y(t)) \\ Y(0) &= \phi(s, X) \end{aligned}$$

  * Recall that $W(t) = \phi(t, X_0)$ solves

$$\begin{aligned} Y'(t) &= F(Y(t)) \\ Y(0) &= X_0 \end{aligned}$$

  * In particular, for $X_0 = \phi(s, X)$, $W(t) = \phi(t, \phi(s, X))$ solves

$$\begin{aligned} Y'(t) &= F(Y(t)) \\ Y(0) &= \phi(s, X) \end{aligned}$$

  * By the uniqueness theory, $W(t) = Y(t)$

$$\phi(t, \phi(s, X)) = \phi(t + s, X)$$

– Integral formulation for flow

$$\phi_t(X) = X + \int_0^t F(\phi_s(X))\, ds$$

- All dynamical systems are flows of ODEs:

  – Given $\phi_t(X)$,we compute $F(X)$

$$\frac{\partial}{\partial t}\phi_t(X)\,|_{t=0} = F(X)$$

  – Since $\phi_{t+s}(X) = \phi_t(\phi_s(X))$

$$\frac{\partial}{\partial t}\phi_{t+s}(X) = \frac{\partial}{\partial t}(\phi_t(\phi_s(X)))$$

  – At $t = 0$,

$$\frac{\partial}{\partial t}\phi_{t+s}(X) = \frac{\partial}{\partial s}\phi_s(X)$$

$$\frac{\partial}{\partial t}(\phi_t(\phi_s(X))) = F(\phi_s(X))$$

  – so $\phi_s(X)$ is a solution of $Y' = F(Y)$

Example 1: Use the Picard iteration to solve

$$X' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X,\ X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Sol: Recall that the solution is

$$X = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

We now generate the Picard sequence:

$$U_{n+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U_n(s)\, ds$$

and show it converges to X.

- Jacobian matrix of mappings $F : R^n \to R^n$

$$F(X) = \begin{pmatrix} f_1(x_1, ..., x_n) \\ \cdots \\ f_n(x_1, ..., x_n) \end{pmatrix}, \quad DF(X) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \nabla f_1 \\ \cdots \\ \nabla f_n \end{pmatrix}$$

Example 2 (a) For $F(X) = X$ in R^n, $\quad DF(X) = I$
(b) In R^2, $F = (x_1 x_2, x_1^2)$

$$DF = \begin{pmatrix} x_2 & x_1 \\ 2x_1 & 0 \end{pmatrix}$$

- Some formulas: $DF(\phi) = DF \cdot D\phi$
- Differentiate DE

$$\frac{\partial \phi(t.X)}{\partial t} = F(\phi(t.X))$$

with respect to X, we see $D\phi(t.X) = D\phi_t(X)$ Solves

$$\frac{\partial D\phi(t.X)}{\partial t} = DF(\phi(t.X)) D\phi(t.X)$$

i.e., each column of $D\phi(t.X)$ solves

$$U' = DF(\phi(t.X)) U$$

or

$$D\phi_t(X) = I + \int_0^t DF(\phi_s(X)) D\phi_s(X) ds$$

- The Variational Equation
  - Given $X_{0,}$ let $\phi(t, X_0)$ be the solution of IVP with the initial value X_0
  - Set $A(t) = DF(\phi(t, X_0))$,i.e., the Jacobian matrix of F evaluated at $\phi(t, X_0)$
  - We call the following **Variational Equation** along solution $\phi(t, X_0)$:

$$U' = A(t) U$$

- Variational equation is a (global) linearization along $\phi(t, X_0)$:
  - If $\phi(t, X_0)$ exists for t in $[0, T]$, then so does the entire flow $\psi(t, U_0)$ for Variational equation, i.e.,

$$\frac{\partial \psi(t, U_0)}{\partial t} = A(t)\, \psi(t, U_0)$$

  - Since

$$\frac{\partial D\phi(t.X_0)}{\partial t} = DF(\phi(t.X_0))\, D\phi(t.X_0)$$

  - each column of $D\phi(t.X_0)$ is a solution of the variational equation
  - Multiplying both sides by U_0 :

$$\frac{\partial D\phi(t.X_0)\, U_0}{\partial t} = DF(\phi(t.X_0))\, D\phi(t.X_0)\, U_0$$

  - Since $D\phi(0, X_0)\, U_0 = U_0$,

$$\psi(t, U_0) = D\phi(t, X_0)\, U_0$$

  - This means that, if we are able to solve the viational equation, then we can easily find $D\phi(t, X_0)$ since

$$D\phi(t, X_0) = [\psi(t, e_1), \psi(t, e_2), ..., \psi(t, e_n)]$$

    where e_i is the standard basis.
  - If $\phi(t, X)$ is smooth, we expand $\phi(t, X_1)$ in X variable around X_0 :

$$\phi(t, X_1) = \phi(t, X_0) + D\phi(t, X_0)(X_1 - X_0) + O\left(|X_1 - X_0|^2\right)$$

  - So if $X_1 = X_0 + U_0$,then

$$\phi(t, X_1) = \phi(t, X_0) + \psi(t, U_0) + O\left(|X_1 - X_0|^2\right)$$

    This means that if we can solve IVP at initial value X_0 to get $\phi(t, X_0)$, then we can calculate $A(t) = DF(\phi(t, X_0))$ and thus to calculate the solution $\psi(t, U_0)$ for the viational equation since it is linear. Then, we can approximate other IVP for nearby initial value $X = X_0 + U_0$ by

$$\phi(t, X_0) + \psi(t, U_0)$$

- In general (with less smoothness condition), let $X(t) = \phi(t, X_0)$, $U(t) = \psi(t, U_0)$, then $X(t) + U(t)$ is an approximation for $\phi(t, X_0 + U_0)$ of order higher than one:

$$\frac{|X(t) + U(t) - \phi(t, X_0 + U_0)|}{|U_0|} \to 0$$

  as $|U_0| \to 0$. The convergence is uniform in t.
- In particular, if X_0 is an equilibrium solution, i.e., $\phi(t, X_0) = X_0$, then

$$A(t) = DF(\phi(t, X_0)) = DF(X_0)$$

  Variational equation is

$$U' = DF(X_0) U$$

  Its solution $\psi(t, U_0)$ is such that

$$\phi(t, X_0) + \psi(t, U_0) = X_0 + \psi(t, U_0)$$

  is an approximation solution for $\phi(t, X_0 + U_0)$. So sometime we call the variational equation linearization.

Example 3 $x' = x^2$.Solution is

$$\phi(t, x_0) = \frac{x_0}{1 - x_0 t}$$

So

$$D\phi(t, x_0) = \frac{1}{(1 - x_0 t)^2}$$

On the other hand, $DF = 2x$. So its variational equation for $x = \phi(t, x_0)$ is

$$u' = 2\phi(t, x_0) u = \left(\frac{2x_0}{1 - x_0 t}\right) u$$

The solution of this variational equation with initial data u_0 is

$$\psi(t, u_0) = \frac{u_0}{(1 - x_0 t)^2}$$

Apparently, this function is defined as long as $t < 1/x_0$, the same as for $\phi(t, x_0)$.

For $x_1 = x_0 + u_0$,

$$\begin{aligned}
&\phi(t, x_0) + \psi(t, u_0) - \phi(t, x_1) \\
&= \frac{x_0}{1 - x_0 t} + \frac{u_0}{(1 - x_0 t)^2} - \frac{x_1}{1 - x_1 t} \\
&= \frac{x_1 - x_0^2 t}{(1 - x_0 t)^2} - \frac{x_1}{1 - x_1 t} \\
&= \frac{x_1 - x_0^2 t - x_1^2 t + x_0^2 x_1 t^2 - x_1 (1 - x_0 t)^2}{(1 - x_0 t)^2 (1 - x_1 t)} \\
&= \frac{-(u_0)^2 t}{(1 - x_0 t)^2 (1 - x_1 t)}
\end{aligned}$$

So as $u_0 \to 0$,

$$\frac{|\phi(t, x_0) + \psi(t, u_0) - \phi(t, x_1)|}{|u_0|} = \left| \frac{u_0 t}{(1 - x_0 t)^2 (1 - x_1 t)} \right| \to 0$$

- When X_0 is an equilibrium solution, the variational equation becomes an autonomous linear system. We call in this case the linearization.

Example 5: Consider $X' = F(X)$, $F = (x + y^2, -y)$. $X = 0$ is an equilibrium. So along this solution, the linearization is

$$DF(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; U(t) = \begin{pmatrix} x_0 e^t \\ y_0 e^{-t} \end{pmatrix}$$

- Homework: 1ce, 2, 7(for the case (i) $A(t) = diag(\lambda_1(t), ..., \lambda(t))$, (ii) $n = 2$, general matrices that may not be diagonal)

- Homework (additional): Find and solve the variational equations for $X' = F(X)$

  1. $F = (x^2 + xy, x + y^3)$, $X = (-1, 1)$
  2. $x' = x^{4/3}$, $x(t) = 27(3 - t)^{-3}$

- Homework (optional): Write (or find from internet) a numerical program (in any language, such as C, Matlab, Mathematica, etc.) for (i) Euler's method (ii) Runge-Kutta of order 4.