Chapter 6 Higher Dimensional Linear Systems

Linear systems in R^n:

$$X' = AX$$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

- Observations:
  - Let T the the coordinate change matrix such that $T^{-1}AT$ is in canonical form B,then $Y = T^{-1}X$ solves

    $$Y' = T^{-1}X' = T^{-1}AX = \left(T^{-1}AT\right)\left(T^{-1}X\right) = BY$$

  - For block-diagonal matrix

    $$B = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix}$$

    the system $Y' = BY$ is reduced to total of k smaller linear systems

    $$Y_j' = B_j Y_j$$

    $$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_k \end{pmatrix}$$

  - So it suffices to solve $Y' = BY$ for B in the following two forms

    $$(i) \quad \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}_{p\times p}, \quad (ii) \quad \begin{pmatrix} C_2 & I_2 & & \\ & C_2 & I_2 & \\ & & & I_2 \\ & & \ddots & C_2 \end{pmatrix}_{2q\times 2q}$$

  - where

    $$C_2 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Case (i):
  - $Y' = BY$ is

$$\begin{aligned} y_1' &= \lambda y_1 + y_2 \\ &\cdots \\ y_{p-1}' &= \lambda y_{p-1} + y_p \\ y_p' &= \lambda y_p \end{aligned}$$

  - each is a linear first-order DE
  - we start with solving the last equation

$$y_p = c_p e^{\lambda t}$$

  - and substitute it into the one above it:

$$y_{p-1}' = \lambda y_{p-1} + y_p = \lambda y_{p-1} + c_p e^{\lambda t}$$

  - and solve this linear DE:

$$\begin{aligned} \left(e^{-\lambda t} y_{p-1}\right)' = e^{-\lambda t} y_{p-1}' - \lambda e^{-\lambda t} y_{p-1} = e^{-\lambda t} \left(y_{p-1}' - \lambda y_{p-1}\right) &= c_p \\ e^{-\lambda t} y_{p-1} &= c_p t + c_{p-1} \\ y_{p-1} &= \left(c_p t + c_{p-1}\right) e^{\lambda t} \end{aligned}$$

  - In the same manner, we can "move upward" to solve y_{p-2},then $y_{p-3}, \ldots$,till finally solve y_1

- Case (ii):
  - $q = 1,\ B = C_2$. We know from planar system (chapter 3), for $\lambda = \alpha + i\beta$

$$Y = e^{\alpha t} \begin{pmatrix} a\cos\beta t + b\sin\beta t \\ -a\sin\beta t + b\cos\beta t \end{pmatrix}$$

  - for $q > 1$,we write

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_q \end{pmatrix}, \quad Y_j = \begin{pmatrix} y_{j1} \\ y_{j2} \end{pmatrix}$$

- We again start with the last DE and move backwards. Solving the last equation:

$$Y_q' = C_2 Y_q \implies$$

$$Y_q = e^{\alpha t} \begin{pmatrix} a_q \cos \beta t + b_q \sin \beta t \\ -a_q \sin \beta t + b_q \cos \beta t \end{pmatrix}$$

- Substitute into the next one above

$$Y_{q-1}' = C_2 Y_{q-1} + Y_q$$

- This is a planar system of linear nonhomogeneous DEs. it may be solved using the method of "variation of parameters" by looking for solution in the form (optional homework)

$$Y = e^{\alpha t} \begin{pmatrix} a_q (t) \cos \beta t + b_q (t) \sin \beta t \\ -a_q (t) \sin \beta t + b_q (t) \cos \beta t \end{pmatrix}$$

- once we solve this system, we can then move upward to solve for $Y_{q-2}, Y_{q-3}, ..., Y_1$ successively.
- We shall introduce another approach to solve this nonhomogeneous system

- In summary, to solve $X' = AX$,
  1. we first find its canonical form $T^{-1}AT = B$.
  2. next, we solve $Y' = BY$ by solving several subproblems in case (i) and/or case (ii)
  3. Finally, $X = TY$ is the desired solution.

Example 1 Solve $X' = AX$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Example 2 Solve $X' = AX$

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Sol:

$$T=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ B=\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- **The Exponential of A Matrix**
  - Recall that in solving case (ii), we need to solve nonhomogeneous system

    $$Y' = C_2 Y + Y_q(t),\quad Y_q(t) \text{ is a given vector function}$$

  - The method of variation of parameter is used then
  - but better methods are need
- Recall the Taylor series expansion

  $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

  - it is convergent for all x.
  - for diagonal matrix $A = diag(\lambda_1, ..., \lambda_n)$

    $$A^k = diag\left(\lambda_1^k, ..., \lambda_n^k\right)$$

  - So as $N \to \infty$

    $$\sum_{k=0}^{N} \frac{A^k}{k!} = diag\left(\sum_{k=0}^{N} \frac{\lambda_1^k}{k!}, ..., \sum_{k=0}^{N} \frac{\lambda_n^k}{k!}\right) \to diag\left(e^{\lambda_1}, ..., e^{\lambda_n}\right) = e^A$$

- Definition of e^A

  $$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

- Theorem: The above series convergent if we define metric in $L(R^n)$ as, for $A = (a_{ij})$

  $$\|A\| = \max(|a_{ij}|)$$

Example 3. Find e^A if

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

Example 4. Find e^A if

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} = \beta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \beta E$$

Sol: Note that $E^2 = -I,\ E^3 = -E,\ E^4 = I$. So

$$e^A = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$$

Example 5. Find e^A if

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Sol:

$$e^{At} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

Example 6. Guess what is e^A if

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \mu \end{pmatrix}?$$

- Properties of exponential of matrices:

  1. If $B = T^{-1}AT$, then $e^B = T^{-1}e^A T$
  2. If $AB = BA$, then $\exp(A+B) = e^A e^B$
  3. $\exp(-A) = (\exp(A))^{-1}$
  4. If λ is an eigenvalue of A and V is an associated eigenvector, then e^λ is an eigenvalue of $e^A$and V is an eigenvector of e^A associated with e^λ
  5. $\left(e^{tA}\right)' = Ae^{tA} = e^{tA}A$

- Theorem: $e^{tA}X_0$ is the only solution of

$$X' = AX, \quad X(0) = X_0$$

Example7: Find general solutions for $X' = AX$ with A in Example 3-5:

1. In Example 3.

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad e^{tA} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix}$$

   So

$$X = e^{tA} X_0 = \begin{pmatrix} e^{\lambda t} x_0 \\ e^{\mu t} y_0 \end{pmatrix}$$

2. In Example 4.

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}, \; e^{At} = \begin{pmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{pmatrix}$$

   So

$$X = e^{tA} X_0 = \begin{pmatrix} x_0 \cos \beta t + y_0 \sin \beta t \\ -x_0 \sin \beta t + y_0 \cos \beta t \end{pmatrix}$$

3. In Example 5.

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad e^{At} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

   So

$$X = e^{tA} X_0 = \begin{pmatrix} x_0 e^{\lambda t} + y_0 t e^{\lambda t} \\ y_0 e^{\lambda t} \end{pmatrix}$$

- Nonhomogeneous

$$X' = AX + G(t)$$

- Variation of Parameters: Consider solution in the form

$$X = e^{tA} Y$$

  Note that when $G = 0$, $Y(t) =$ constant. Now consider $Y(t)$ is a function. Substituting it into the equation

$$\begin{aligned} LHS &= X' = \left(e^{tA} Y\right)' = \left(e^{tA}\right)' Y + e^{tA} Y' = Ae^{tA} Y + e^{tA} Y' \\ RHS &= AX + G(t) = Ae^{tA} Y + G(t) \end{aligned}$$

So

$$e^{tA}Y' = G(t), \quad \text{or} \;\; Y' = e^{-tA}G(t)$$

and

$$Y = X_0 + \int_0^t e^{-As}G(s)\,ds$$

- Solution of nonhomogeneous IVP is

$$X = e^{tA}\left(X_0 + \int_0^t e^{-As}G(s)\,ds\right)$$

Example 8: Find solution for systems in case (ii) of canonical forms

$X' = C_2X + G(t), \quad G(t) = (g_1(t).g_2(t))$ is a given vector function

Sol: To find e^{tC_2}, we first try:

$$C_2 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \; C_2^2 = \begin{pmatrix} \alpha^2-\beta^2 & 2\alpha\beta \\ -2\alpha\beta & \alpha^2-\beta^2 \end{pmatrix}$$

$$C_2^3 = \begin{pmatrix} \alpha^3-3\alpha\beta^2 & -\beta^3+3\alpha^2\beta \\ \beta^3-3\alpha^2\beta & \alpha^3-3\alpha\beta^2 \end{pmatrix} \quad \cdots$$

It seems not so easy! We try something else. Notice that $X = e^{tC_2}X_0$ is the solution of

$$X' = C_2X, \quad X(0) = X_0 = \begin{pmatrix} a \\ b \end{pmatrix}.$$

On the other hand, we know the solution of this planar system is

$$X = e^{\alpha t}\begin{pmatrix} a\cos\beta t + b\sin\beta t \\ -a\sin\beta t + b\cos\beta t \end{pmatrix}$$

So

$$e^{tC_2}\begin{pmatrix} a \\ b \end{pmatrix} = e^{\alpha t}\begin{pmatrix} a\cos\beta t + b\sin\beta t \\ -a\sin\beta t + b\cos\beta t \end{pmatrix}$$

In particular, if we choose $a = 1, b = 0$, and $a = 0, b = 1$, respectively, then we shall see

$$e^{tC_2} = e^{\alpha t}\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}.$$

Recall

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

So

$$\begin{aligned} e^{-tC_2} &= \left(e^{\alpha t}\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}\right)^{-1} \\ &= e^{-\alpha t}\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}^{-1} = e^{-\alpha t}\begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix} \end{aligned}$$

and thus

$$\begin{aligned} X &= e^{tC_2}\left(X_0 + \int_0^t e^{-C_2 s} G(s)\, ds\right) \\ &= e^{\alpha t}\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}\begin{pmatrix} x_1 + \int_0^t \left(g_1(s)\cos\beta s - g_2(s)\sin\beta s\right) e^{-\alpha s} ds \\ x_0 + \int_0^t \left(g_1(s)\sin\beta s + g_2(s)\cos\beta s\right) e^{-\alpha s} ds \end{pmatrix}. \end{aligned}$$

Example 9: Harmonic oscillators

$$x'' + x = \cos\omega t$$

or

$$X' = AX + \begin{pmatrix} 0 \\ \cos\omega t \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Recall that

$$\begin{aligned} e^{At} &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \\ e^{-At} &= e^{A(-t)} = \begin{pmatrix} \cos(-t) & \sin(-t) \\ -\sin(-t) & \cos(-t) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \int_0^t e^{-As} G(s)\, ds &= \int_0^t \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}\begin{pmatrix} 0 \\ \cos\omega s \end{pmatrix} ds \\ &= \int_0^t \begin{pmatrix} -\sin s\cos\omega s \\ \cos s\cos\omega s \end{pmatrix} ds = \frac{1}{2}\begin{pmatrix} \frac{\cos(\omega+1)t}{(\omega+1)} - \frac{\cos(\omega-1)t}{(\omega-1)} + \frac{2}{\omega^2-1} \\ \frac{\sin(\omega+1)t}{(\omega+1)} - \frac{\sin(\omega-1)t}{(\omega-1)} \end{pmatrix} \end{aligned}$$

So

$$X = e^{tA}X_0 + e^{tA}\int_0^t e^{-As}G\left(s\right)ds$$

- Homework: 1(h), 4, 6, 7, 12(c)(g)(j)